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XIX. *On the Conjugate Positions of two Circular Coils of Wire.* By W. GRANT, *Assistant in the Physical Laboratory, University College, London*.*

[Plate XVII.]

WHILE recently engaged on some experiments on induction, I observed certain circumstances which I had not before noticed, and which seemed deserving of further attention. I was therefore led to inquire a little more closely into these matters; and although the investigation is by no means full or complete, I have obtained one or two results which I thought I might venture to lay before the Physical Society. The apparatus used in these experiments consisted, as at first arranged, of two coils of copper wire, one of which was connected in circuit with a battery of three Leclanché cells, and with a microphone which was actuated by a watch, while the other was connected with a telephone, in order that the induced currents, while passing through it, might render audible the beating of the watch which was used as the source of sound.

A modification of this arrangement was afterwards tried, a Grove's battery of twelve cells being substituted for the Leclanché battery, and a key being used for making and

* Read June 28th, 1879.

breaking the circuit. This was done in order to obtain greater inductive effects between the coils than could be obtained from the variations in the strength of the current which were caused by the action of the microphone. It was found, however, that with a little care in the adjustment of the coils, one cell gave sensibly as great an effect in the telephone as twelve cells; in subsequent experiments, therefore, the Grove's battery was discarded, and that of Leclanché again resorted to.

Now if two similar coils, connected as above described, are arranged with their planes parallel and their axes coincident, it is found that they may be separated to a considerable distance before the sounds which are heard in the telephone on making and breaking the circuit are obliterated. But it is also found that if the planes of the coils are kept parallel, the one in connexion with the telephone (that is, the secondary coil) may be placed in certain positions in the neighbourhood of the primary coil, and even in contact with it, without sounds being heard in the telephone. This happens when the mutual inductive effect between the two coils becomes zero; and when they are so placed as to fulfil this condition, they are said to occupy conjugate positions relatively to each other.

With the first arrangement of apparatus it was possible to place the coils so as to get complete silence in the telephone. With the powerful current from twelve Grove's cells and the key for making and breaking the circuit, however, the silence is not absolute; but in the positions which give a minimum of sound the sound is very faint, being just audible and no more. This faint sound may perhaps be accounted for partly because the different convolutions of wire in the secondary coil experience slightly different inductive effects from the primary one, and partly because it is difficult to adjust the positions of the coils with any great degree of accuracy without having special arrangements for the purpose.

Now it was found that the various conjugate positions in which the secondary coil could be placed in the neighbourhood of the primary one were situated in a path along which it could be moved either towards or away from the primary coil without sounds being heard in the telephone, but that

with a slight deviation from this path to either side the sounds were again heard.

In order to ascertain whether the direction of the currents in the secondary circuit was reversed when the coil was moved from one side of the path to the other, a delicate reflecting galvanometer was substituted for the telephone, and the position of the coil so adjusted that on making and breaking the circuit no deflection of the galvanometer was observed. The coil was now moved slightly away from this position, say, towards the right; and the direction of the deflection of the galvanometer on making contact was noted, that on breaking being, of course, in the opposite direction. The coil was now moved towards the left to the other side of the path, and the direction of the deflections again observed; and it was found that they were now reversed. We may therefore infer that this path (which, if it could be fully traced, would of course constitute a surface of revolution about the axis of the primary coil) divides space into two regions, in one of which the inductive action of the primary coil has the opposite direction to what it has in the other.

This path appeared to be slightly curved; and it seemed as if a part of it might very readily be traced. The part which appeared to be best suited for this purpose was that along which the secondary coil has to pass while being moved away from contact with the primary one parallel to it to a position at some distance from it, as here the inductive effect is greatest, and therefore any deviation of the coil from the proper position in the path is most easily detected. As the coils are further separated, however, the position of the path becomes more difficult to trace, until at last we lose it altogether.

In order, then, to trace a curve which would represent this path, it was necessary to find several points in it whose positions could afterwards be accurately laid down. This was done by fixing the secondary coil in several positions successively and determining the position of a certain point in it with relation to certain fixed objects, by measurements which were afterwards used as abscissæ and ordinates in tracing the curve. These measurements were taken in inches; and their values are given in the annexed Table, where the columns

headed x and y are those of abscissæ and ordinates respectively.

x.	y.	x.	y.
0·625	4·0	4·5	7·5
1·0	4·19	5·0	8·12
1·5	4·56	5·5	8·75
2·0	5·0	6·0	9·37
2·5	5·5	6·5	10·0
3·0	5·94	7·0	10·62
3·5	6·44	7·5	11·25
4·0	7·0	8·0	11·87

No special arrangement was used to adjust the parallelism of the coils, and only one measurement was taken for each number; hence the irregularity in the increase of the ordinates. The point whose position was determined in each case was the centre of the plane of the secondary coil; and that is the point which is situated in the curve when silence is maintained in the telephone.

The curve (Pl. XVII.) is that found in this way; and it represents the path which the selected point of the coil has to follow in order that silence may be maintained in the telephone. C_1 and C_2 are sections of the primary and secondary coils respectively; C'_2 and C''_2 represent the secondary coil in two other conjugate positions. The lines A, A and P, P represent the axis and plane of the primary coil. The points a, b, c, &c. are the intersections of the abscissæ and ordinates, and represent the successive positions occupied by the selected point of the secondary coil when the measurements were taken by means of which the curve was traced. As the coils became further separated, however, the position of the curve became less distinct; and so no attempt was made to trace it further.

If, now, we suppose the curve to rotate round the axis of the primary coil, a surface will be generated of which it is a section; and if we observe the conditions necessary for placing the secondary coil in the curve in the proper position for silence, we may place it in any part of the surface with a like result.

The reason why we are enabled to trace a curve in this way will be found by referring to the lines of force due to a circular current. These lines are represented by closed curves surrounding the section of the wire through which the current

flows; and they are given in Prof. Clerk Maxwell's work 'On Electricity and Magnetism,' vol. ii. pl. 18. If we draw tangents to them parallel to the plane of the circular current, it will be found that the points where they touch are situated in a curve somewhat similar to that which we have found by experiment. The two curves, however, will not be found to coincide exactly, except in the case where the secondary coil does not enclose a space—that is to say, when its diameter is infinitely small. With respect to the curve drawn through the points of contact of the tangents to the lines of force, it will be seen that the direction of all these lines between the curve and the axis of the circular current is away from, and that their direction on the other side of the curve is towards the plane of the circular current: hence on opposite sides of the curve their tendency is to produce currents in opposite directions.

If the curve is now supposed to revolve round the axis of the circular current, all lines of force enclosed by the surface generated will tend to produce currents in one direction, while all lines outside the surface will tend to produce currents in the opposite direction. Therefore, when the secondary coil is so situated with respect to this surface that as many lines of force pass through it in one direction as in the other, the resultant inductive effect on it will be zero; and this will be the case when it occupies any of the conjugate positions*.

It is evident from this, therefore, that we may combine the coils in several ways for the suppression of inductive effects:—first, by placing them close together face to face with their axes coincident, and so arranged that one of them may be moved across the face of the other parallel to their planes till a balance is obtained; secondly, by placing them at some distance apart with their planes parallel and their axes coincident, and so arranged that if their planes are vertical each of them

* In what precedes, the planes of the coils have been always supposed to be parallel to each other; but it evidently follows from the reasoning here indicated that, if any set of parallel tangents be drawn to the lines of force and a curve be traced through the points of contact, an infinitely small coil would experience no inductive effect if it were placed with its centre anywhere in this curve, and with its plane parallel to the given set of tangents.

may be made to rotate round its vertical diameter ; then if they are joined together when their axes are coincident, and combined like parallel rulers, they may be made to rotate together until a balance is obtained. With regard to this combination, it may be observed that the greatest inductive effect occurs when the planes of the coils are at the greatest distance from one another—and that as the planes approach, this effect gradually diminishes, until, when they are still at some distance, it becomes nothing.

Another, and perhaps more convenient, way of combining them is to place them, as in the last case, with their planes parallel and their axes coincident, the distance between them being equal to, or a little greater than, the radius of either coil: then, if their planes are vertical, we may fix one of them in that position ; and if the other is capable of rotating round its vertical diameter, it will be found that when it has rotated through 90° (that is, when the planes of the coils are at right angles) the inductive effects in the secondary circuit have ceased. If the coil is made to rotate through a few degrees to one side of this position, the currents induced in it will be in a certain direction ; and if it is rotated to the other side, their direction will be found to be reversed.

As with either of these combinations we could pass from sound to silence, some experiments were made in order to compare the rate of diminution of the induced currents with the movements of the coils in passing from a maximum to a minimum of inductive effects.

For this purpose the coils were placed with their faces in contact and their axes coincident, the secondary one being joined in circuit with a reflecting galvanometer. In this position five observations were taken and the mean recorded. They were now separated until their planes were an inch apart, and a mean of five observations again taken ; and this process was repeated at intervals of half an inch till the distance between them was increased to five inches.

They were now arranged as in the first combination, their faces being in contact during all the experiments ; and while their axes were coincident, five observations were taken and the mean recorded. One of them was now slid over the other, the faces being still in contact, through a distance of half an

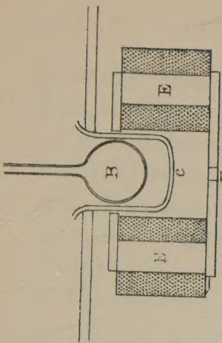


Fig. 2.

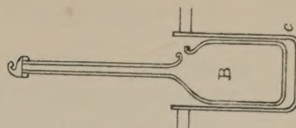


Fig. 4.

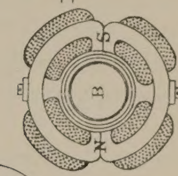


Fig. 3.

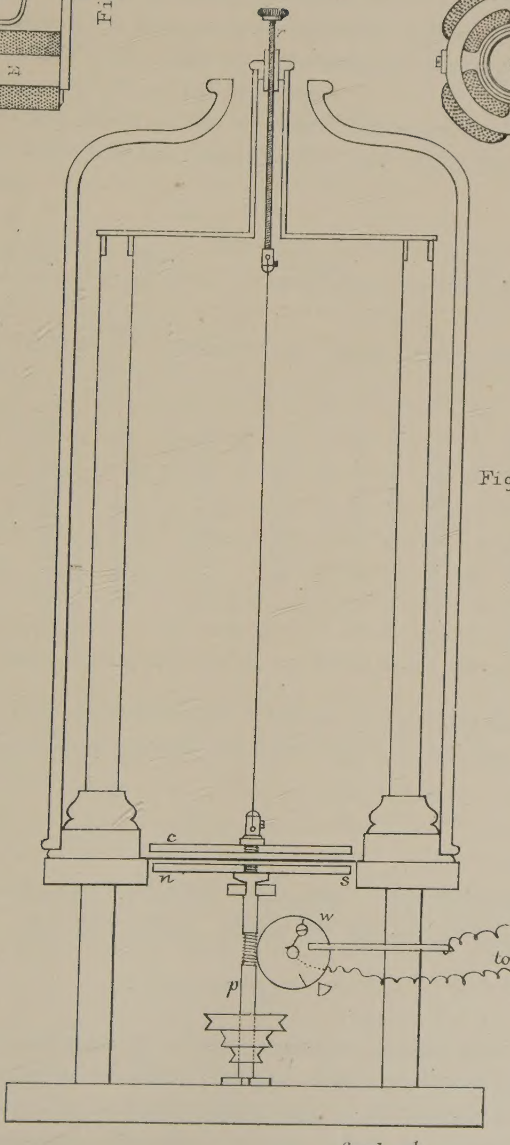


Fig. 1.

Scale $\frac{1}{3}$.

inch and a mean recorded as before; and this process was repeated at intervals of half an inch till a balance was established.

The second and third combinations were treated in the same manner, the coils being moved by steps of 10° at a time, and readings taken till a balance was obtained; and as the deflections were small in all cases, they were taken as being proportional to the strength of the currents.

The numbers given in the annexed Table are those found in the way indicated, the mean in each case being that of five experiments.

Axes of coils coincident.		First combination.		Second combination.		Third combination.	
Distance of planes.	Mean.	Distance of axes.	Mean.	Angle.	Mean.	Angle.	Mean.
0·625	88	0	88	0°	8	0°	17
1·0	77	0·5	84	10	7	10	17
1·5	52	1·0	70	20	7	20	17
2·0	38	1·5	57	30	6	30	16
2·5	28	2·0	40·6	40	5	40	16
3·0	22·5	2·5	30·8	50	4	50	15
3·5	17	3·0	20	60	3	60	12
4·0	13·4	3·5	10	65	0	70	9
4·5	10	4·0	0	...	...	80	5
5·0	8	...	...	...	...	85	2
						90	0

Note.—I may state here that I intend to continue this subject, and, when time permits, to trace some of the curves of equal induction.

XX. *On Magneto-Electric Induction.* *By* Frederick Guthrie *and* C. V. Boys, *Assoc. R. School of Mines.*—Part I.

[Plate XVIII.]

It is well known that the electric currents caused in a conductor by relative motion between that conductor and a magnetic pole are in such a direction as to impose a drag upon such motion. An ideal friction (that is, one without

recoil) is called forth; and, accordingly, in the experiments sometimes called Arago's such current-friction exhibits itself in the pursuit of the moving element by the one which is free to move, though originally at rest. Again, the same current-friction appears in the damping of the swing of the compass-needle by neighbouring metallic masses, and by the evolution of heat in the metallic disk revolving between magnetic poles.

It occurred to one of us that such pursuit as occurs in Arago's experiment might be made use of with advantage to measure the rate at which machinery is moving. For we have merely either (1) to connect a revolvable magnet with the machinery, and measure the angle through which it turns a copper plate in its near neighbourhood, which is restrained by increasing torsion, spring, or weight moment, or (2) to connect a copper plate with the machinery so as to revolve in the neighbouroood of a magnet restrained by similar means or by the earth's directive magnetism. The latter plan indeed might seem to have the obvious advantage that, as in the tangent-galvanometer, the amount of the effect is independent of the strength of the magnet. But this advantage is probably more than counterbalanced by the limitation of the angle of deflexion to less than a right angle, whether the magnet swing in a horizontal or a vertical plane.

We have in the following experiments adopted the plan of making a permanent steel or electromagnet revolve in the neighbourhood of the conductor, and governing the motion of the latter by the torsion either of a hair spring of a watch or fine platinum wire.

The accuracy of measurement which can thus be reached, and the greatness of the effects, led us to the hope that the electrical conductivity of liquids might be detected and measured by the same means—a result much to be desired, because, while the effect is due to the passage of currents in the liquid, such passage is wholly unaccompanied by electrolysis and its attendant incubus, polarization.

Before undertaking this latter investigation we have re-examined experimentally the results previously obtained with metallic conductors by others; and as our method differs somewhat from those previously employed, and our results are more extended and in some cases at variance with those of

former experimenters, we venture to lay them before the Society *.

Our own Experiments.

Our experiments were mapped out as follows :—

Other things being the same,—

I. Vary rate of rotation.

II. Vary distance between the elements.

III. Vary diameter of disk.

IV. Cut metallic disk.

V. Vary thickness.

VI. Vary nature of metal of disk.

VII. Examine liquids in view of the determination of their conductivities.

I. *The Relation between Velocity and Deflexion.*—The apparatus used for establishing this relationship is, in its later form, shown in Pl. XVIII. fig. 1 one third the true size, the difference being that the torsion-thread was held by a clip attached directly to a glass shade instead of to the sliding arrangement shown.

The motive power was a Froment's electro-magnetic engine. The elastic band from the engine passed round one or other of the wheels on the vertical spindle, p, carrying a pair of magnets, $n\,s$, each $3\frac{3}{16}$ inches long $\times 1\frac{1}{8} \times \frac{1}{8}$, placed with their similar poles together. A screw on the spindle geared with a worm-wheel, w, with ninety-seven teeth. The time of a revolution of this could be measured either by watching a mark on the wheel pass a fixed pointer, or by its making a contact every revolution so as to sound a bell. The torsion-thread used was the hair spring of a watch, to which was attached the copper disk by means of a screw clip. The speed was altered by placing the driving-band on one or other of the pulleys on the vertical shaft, or on the Froment engine, or by altering the strength of the battery driving it, or by means of a friction-break.

The disk was graduated into degrees on its upper face, and the reading made by keeping a vertical edge, the eye, and

* At the recommendation of the Publication Committee of the Physical Society, we omit an historical summary of the previous investigations of others.

the torsion-thread in one plane, and noting the division or part of one intersected by that plane, so as to avoid parallax.

When the deflexion of the disk exceeds 20° or 30° the wire takes a "set," especially if it is kept long at its maximum twist. It appears completely to recover itself in time; but such recovery is at last so slow that it is preferable to redetermine the zero immediately after each experiment, and to allow for this, assuming that at the moment of observation (that is, when the disk is at its greatest excursion) the set in the wire is the same as it is immediately after when the zero is redetermined. When the deflexion is considerable it is impossible to keep it absolutely constant; accordingly the mean was taken of several maxima and minima, and the rate was registered after each set of observations. The means of the means of deflexion and rate were finally taken. When the rate is high and the deflexion considerable, the disk begins to swing, the axis describing a cone, until reading becomes impossible. This is possibly due to the vertical repulsion between the moving magnet and the disk not being quite symmetrical. In Experiment 1 all the data are given; in Table I. the means only are given.

Experiment 1.

O=0.

Maximum deflexion.	Minimum deflexion.
368°	363°
365	362
366	359
367	357
367	359

Index-wheel went five times round in 49 seconds.

364	359
363	360
364	361

Index-wheel went five times round in $49\frac{1}{4}$ seconds.

366	357
365	355

O=+2°.

The index-wheel going round once corresponds to ninety-seven revolutions of the magnet.

Allowing 2° for the effective set, we have a deflexion of 360°·25 caused by a velocity of 9·825 revolutions a second. In Table I. in the first five lines the rate-observations were simultaneous with the angle-observations; in the latter six they were taken as in the example given: each result is the mean of about eight observations.

TABLE I.

Angular-velocity, rotations per second.	Angular torsion.
0·258	9°·7
0·527	19·3
1·672	60·5
1·702	61·3
2·517	91·2
2·900	106·9
5·374	193·9
7·886	285·9
8·186	303·9
9·877	360·3
23·307	840·0

It appears from these numbers and from their graphic representation in line 1, fig. A, that the torsion varies directly

Fig. A.

as the rate. So exactly is this the case that we may in future with perfect confidence reduce from one rate to another or to

any arbitrary common rate when the rate varies between experiments, to test the effect of variation of other kinds. And we have, if the magnets and torsion keep constant, a very exact and not inconvenient instrument for measuring the rate of rotation of machinery.

II. *The Relation between Distance and Deflexion.*—Two series of experiments were made to determine this relationship. We need only reproduce here the longest and most complete. The apparatus used was that shown in fig. 1, Plate XVIII., which is drawn to a scale one third true size.

The copper disk, *c*, was replaced by one of ebonite, to the lower surface of which the metal plate could be attached by a drop of weak gum. The copper disk used was only 0·27 millim. thick ; so that all its parts might be considered at the same distance from the magnet. The magnet used was $3\frac{5}{16} \times \frac{1}{2} \times \frac{3}{16}$ inch. The upper end of the torsion-thread was attached by a screw-clip to the brass rod, and graduated in millimetres. This could be slid up and down, so as to vary the distance between the magnet and the disk. To get the absolute distance, the mica screen between them was removed, and a piece of copper exactly 1 millim. thick laid on the magnet, and the disk lowered till it just began to rest on the copper. Then the copper was replaced by the mica, and the first experiment made. Then the disk was raised 1 millim. by means of the graduated rod *r* for each succeeding experiment.

The ebonite disk and central clamp had been found to be unaffected by the fastest speed used. In order to avoid extravagant torsion, the rate of rotation was made less when the distance was small.

In Table II., column D shows the distance in millimetres between the upper surface of the magnet and lower surface of the copper disk ; A shows the observed angles of depression ; T shows the time in seconds for one revolution of the worm-wheel ; A′ is the product of A and T. Since T is inversely and A is directly as the rate, A′ shows the pure effect of distance; it would be a constant if the distance did not vary. Each of the experimental numbers given, whether of angle or time, is the mean of never less than three observations. These observations of angle and time were simultaneous.

TABLE II.

D.	A.	T.	A′=A×T.
1	106°·8	13·7	1463·2
2	78·0	13·7	1068·8
3	75·6	10·2	771·1
4	57·0	10·1	575·7
5	68·3	6 5	443·9
6	50·9	7·2	366·5
7	55·1	5·3	292·0
8	43·3	5·3	229·5
9	36·3	5·3	192·4
10	{45·1 {44·6	{3·6 {3·6	{162·4 {160·6
11	36·1	3·7	133·6
12	30·5	3·7	112·8
13	25·6	3·5	96·6
14	21·0	3·6	75·6
15	17·9	3·4	60·9
16	15·5	3·5	54·25
17	13·3	3·6	47·88
18	11·6	3·5	40·60
19	10·4	3·5	36·40
20	{9·0 {10·0	{3·5 {3·6	{31·50 {36·00
21	8·0	3·5	28·00
22	7·1	3·5	24·85
23	6·2	3·5	21·70
24	5·2	3·5	18·20
25	4·8	3·4	16·32
26	4·6	3·4	15·64
27	3·5	3·6	12·60
28	3·5	3·6	12·60
29	3·0	3·6	10·80
30	2·6	3·6	9·36

The second numbers opposite distances 10 and 20 millims. were obtained after the completion of the set of experiments, by resetting the plate to the proper distances, to see if there had been any great change in the power of the magnets or the torsion of the thread.

On plotting out the curve of distance we find line 2, fig. A (p. 131).

III. *Variation in the Diameter of the Disk.*—For the study of this effect the same apparatus was used again, the only difference being that the magnet was only 2 inches long; and a series of disks of copper were cut of the same thickness as that used for finding the law of distance. They were held in the same way—by a drop of weak gum on the under surface

of the ebonite disk. The distance was kept constant, namely 3 millims. In Table III., column Diam. gives the diameter of the disk in inches, column A the corrected mean deflexion of about four observations, column T the time of one revolution of the worm-wheel, got by taking the time of ten or twenty turns, during which the angles are read ; and column A′ is equal to A × T, and represents the torsion for constant velocity.

Table III.

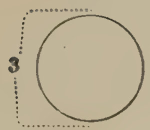
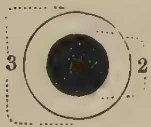
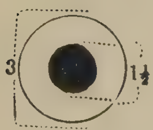
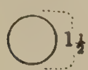
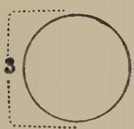
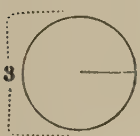
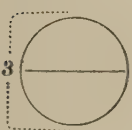
Diam.	A.	T.	A′=A×T.
1·00	0°·3	2·53	·759
1·25	0·9	2·79	2·511
1·50	2·6	2·82	7·332
1·75	6·5	2·83	18·395
2·00	{ 10·7 12·5	{ 3·99 3·90	{ 42·69 48·75
2·25	{ 21·5 21·4	{ 4·02 4·09	{ 86·43 87·53
2·50	30·8	4·03	124·1
2·75	38·7	4·06	157·1
3·00	46·1	4·01	184·9
2·25	50·9	4·00	203·6
3·50	50·0	3·90	218·4

The curve 3 in fig. A is the graphic representation of the above relationship. As was to be expected, there is a point of contrary flexure at about the region where the disk has a diameter equal to the length of the magnet.

IV. *Effects of Cuts in Disks.*—The same apparatus was used, and copper disks 3 inches in diameter, of the same thickness and at the same distance from the 2-inch magnet. In Table IV., A is the observed angle, T the observed time of rotation, and A′ is the product of A and T, and is proportional to the force for a constant rate.

The attached figures represent the way in which the disks were cut.

TABLE IV.

		A.	T.	A′ = A × T.	
1.		44·0	4·03	177·3	
2.		2·8	4·07	11·4	53·4
3.		9·6	4·37	41·95	
4.		16·0	4·31	68·96	77·0
5.		1·8	4·44	8·0	
6.		39·9	4·42	176·4	
7.		37·6	4·35	163·6	
8.		33·0	4·32	142·6	
9.		30·6	4·24	129·7	
10.		26·7	4·24	113·2	

In this Table the dark spaces in 2 and 4 mean absence of metal. The sum of the torsional effects of 2 and 3 or of 4 and 5 is equivalent to that of the disk in 1 having a concentric circular cut 2 inches or $1\frac{1}{2}$ inch in diameter respectively—from which it will be seen that a concentric circular cut interferes with the induced currents far more than radial cuts of equal or even greater length. Experiment 6 is a repetition of experiment 1. The effect of the radial cuts in 7, 8, 9, and 10, though agreeing in kind with the experiments of Babbage and Herschel, is far less in amount; and this is noteworthy, as in their experiments the disks of lead were not cut completely to the centre, but a small space was left to unite the different parts together; while in our experiments the copper disk was cut completely through, the different parts being held in position by the ebonite plate above.

Taking the torsional effect of an uncut disk as 100, the following numbers show the effect of disks cut as in 7, 8, and 10, as obtained by

	7.	8.	10.
Babbage and Herschel . . .	83·2	72·5	44·8
Us	92·8	80·7	64·2

V. *Effect of Thickness.*—The examination of the effect of, thickness is experimentally involved in that of distance, because, in increasing the thickness of a plate, it is impossible with our data to avoid change of distance of plane of effective mass. We have thought it best to employ for this purpose very thin disks cut out of a uniform sheet, and stuck with thin gum under the ebonite disk, at first one, then two, and so on up to six. Tinfoil was used 0·13 millim. thick, and 3·625 in. in diameter.

TABLE V.

No. of foils.	A.	T.	$A'=A\times T$.	Δ A'.
1.	4·9	22·7	111·2	111·2
2.	10·5	21·3	223·6	112·4
3.	15·9	21·3	338·7	115·1
4.	22·0	21·3	468·6	129·9
5.	26·6	22·3	593·2	124·6
6.	30·0	24·6	738·0	144·8
1*a*.	6·3	23·2	146·2	

The column $\Delta A'$ shows the increase in effect due to each

foil; it also shows that the lower foils, being nearer the magnet, are more effective than the upper ones. A single foil 1*a* was afterwards hung at the distance of foil no. 6, and gave the result 146·2, agreeing with 144·8. The effect, then, of disks may be taken as directly proportional to their joint thickness, the mean distance remaining the same. There is probably a misreading with the four disks.

VI. *Effect of Nature of Metal.*—Sheets of copper, zinc, tin, brass, and lead having been rolled between the same rollers, were found to have the thickness as under :—

	millim.
Copper	0·89
Zinc	0·87
Tin	0·71
Brass	0·93
Lead	0·76

Disks 2·875 inches in diameter were cut out of the sheets, and fastened in the usual way by a drop of gum to the under surface of the ebonite disk. The results are given in Table VI.

TABLE VI.

Metal.	A.	T.	$A' = A \cdot T.$	$\frac{A'}{t}$	$r.$	$\frac{A' \times r}{t}$
Copper...	78·5	21 8	1711·3	1923	·119	229
Brass ...	28·8	23·0	662·4	712	·423	301
Zinc ...	31·4	23·0	722·2	830	·326	271
Tin ...	13·0	23·3	302·9	427	·669	286
Lead ...	8·4	23·0	193·2	254	1·109	281

The column headed $\frac{A'}{t}$ is obtained by dividing the numbers in the previous column by the thickness of metal used. The numbers in the column r, give the resistance, in ohms of 100 inches, of wire drawn from the same metal that was rolled into sheets, and ·79 millim. in diam. These two columns multiplied together give the last column, which should be a constant if the deflexion is proportional to the conductivity. Now these numbers range from 229 to 301, which seems rather a wide divergence ; but in this case there are many considerations to be taken into account, in each of which errors may easily

creep in, especially in the assumption that the specific conductivity of the same metal is the same whether it be rolled or drawn. The approximation to equality, obtained in the numbers in the last column, is all that could be expected.

It appears from the above experiments in I., II., III., IV., V., VI., that the torsion varies directly with the moment of the current.

VII. *Examination of Liquids.*—The electro-magnet seen in vertical section in Pl. XVIII. fig. 2, one third true size, was made to screw onto a vertical axis, so as to revolve in a horizontal plane. A 5-in. sulphuric-acid dish was held just above it by a separate stand to avoid any effects that vibration might cause; and into this dish the liquid to be tested could be put; a cover was also put over the dish. Mercury was first tried; and it soon began to rotate in the same direction as the magnet. Dilute sulphuric acid of the best-conductivity strength was next experimented on: a wood float that had been soaked in paraffin to make it float in the middle of the vessel served for an index to show motion of liquid. The float followed the magnet at the rate of about 15° an hour, but was always rather uncertain in its movement. The only noticeable feature of this experiment was, the fine sulphate-of-lead powder which had settled during the night arranged itself during the experiment in fine concentric rings, the circle immediately above the poles of the electro-magnet being bare. A solution of sulphate of copper showed no indications whatever.

As this method did not seem at all promising, the apparatus shown in vertical section in fig. 2 was next used. A glass bulb B was filled with the same sulphuric acid that was used in the last experiment, and hung by a silk thread in a small beaker *c* which acted as a screen; there was also a screen above. The electro-magnet *e* was made to revolve before the current was turned on, to see if there was any mechanical effect; but there was none at all; as soon, however, as the wires were connected with the battery the bulb B began to swing round, and stopped at about 180°. Here there was an unmistakable effect obtained with a non-metallic liquid conductor.

If, then, the torsion is proportional to the conductivity of the liquid in the vessel, which it undoubtedly is, we have the means of measuring the conductivity of liquids without using elec-

trodes, and therefore without polarization. To see how the method worked, the preliminary apparatus shown in figs. 3 and 4 was constructed. N S is a double electro-magnet made from iron bar 1 in. wide and ¼ thick, bent as shown in horizontal section in fig. 3, so that the two poles are inside: with this arrangement the lines of force which intersect the cylindrical vessel B are almost straight and parallel. The vessel B was hung in the cylindrical screen *c* by a platinum wire 15 ft. long and 0·002 in. in diameter; and a small mirror was fixed to it for observing the reflexion of a small gas-flame. The liquid could be introduced by a pipette through a small hole in the side. Mixtures of sulphuric acid and water were made in the following proportions by weight, and were tested in the above apparatus:—

	a.	*b.*	*c.*	*d.*	*e.*
Acid	7	6	5	4	3
Water	3	4	5	6	7

They gave the following numbers, which represent the number of divisions on a scale of equal parts, at which the reflexion of the light could be seen when the mirror was in its position of mean deflexion.

a 2·1, *b* 2·6, *c* 5·5, *d* 6·8, *e* 8·9.

These numbers, being obtained only from a provisional apparatus, must not be considered to represent the accuracy of the process, but merely as showing that larger and more perfect apparatus is likely to give good results.

With regard to the validity of this method of determining the conducting-power of liquids a few words may be said. 1st. What will be the action of para- or dia-magnetic liquids apart from conductivity? Iron gives a greater turning effect than copper when placed over a revolving magnet, owing to its very great magnetizability and to its retaining magnetism a perceptible time; but whether liquids can acquire sufficient magnetism and retain it long enough to vitiate the results it is impossible to say; experiment will show. 2nd. Does the torsion of the wire really represent the twisting effect on the liquid *at rest*? It would seem at first that it does not; but a little consideration will show that no error in any way appreciable can be committed on the supposition that it does. The

action is this: the rotating magnets cause the liquid in the vessel to rotate in the same direction; the friction between the liquid and the vessel tends to make the vessel rotate too; but it is restrained by the torsion of the wire. As the inertia of the liquid has to be overcome by a comparatively small force, its velocity will increase very slowly; and were there no friction against the sides of the vessel it would, if only left long enough, equal that of the magnet. But there is friction, which increases with the velocity; and therefore as the velocity increases the acceleration will diminish, and in time the velocity will be constant—that is, when the twisting force and the friction become equal—just as a light body when dropped in air soon attains a constant velocity. This velocity depends on the viscosity of the liquid; and it might be well said that two liquids having the same conductivity, but one viscous and the other mobile, would attain very different constant velocities, and, as the twisting force is directly proportional to the *relative* motion between the liquid and the magnet, that different results would be attained. This is quite true; but the question turns upon the velocity which the liquid attains. If it is itself comparable to the velocity of the magnet, then any changes in it will affect the result but if it is nothing compared with the magnet's velocity, then no error at all can be produced. Now on looking back it will be seen that the rotation of the best-conductivity acid in the 5-in. basin over a revolving electro-magnet turning about 3000 times a minute, was about 15° an hour; in other words, the magnet went about 4,320,000 times as fast as the liquid; and the total possible error on the supposition that the liquid is still, would be $\frac{1}{4320000}$ of the whole amount; therefore it may be safely neglected.

It appears, therefore, that the point to be borne in mind is that the torsion must not be measured till the liquid has attained its constant velocity; *i. e.* sufficient time must be allowed. This velocity and this time will be less the greater the friction or viscosity. But the introduction of glass threads, screens, or porous plates to increase the friction would do more harm by introducing uncertain electrical resistances than they could possibly do good: the only device which seems as if it might be of any use would be to make a jelly of the conducting liquid; but this is quite unnecessary.

We are engaged in measuring the conductivity of liquids by means similar to those above shown.

EXPLANATION OF FIG. A (page 131).

Curve 1. The abscissæ are in proportion to the velocities of rotation of a permanent steel magnet beneath a copper disk; the ordinates are in proportion to the angles at which the disk comes to rest. This curve appears to be a straight line, showing that the torsional moment is proportional to the velocity.

Curve 2. The abscissæ are in proportion to the distances between a thin copper disk and the upper surface of a permanent steel magnet revolving beneath; corrected for rate according to curve 1, the ordinates are in proportion to the angles at which the disk comes to rest.

Curve 3. The abscissæ are in proportion to the diameter of a thin copper disk, varying from 1 in. to 3·5 in. in diameter, turned by a revolving permanent magnet beneath, 2 in. long; the ordinates are in proportion to the angular deflexions. The double line shows the secondary observations about the region (the "pole") of the magnet where the effect of variation of diameter should be most marked. About this place there is an alteration in flexure.

XXI. *On the Determination of the Variation of the Thermal Conductivity of Metals with Temperature, by means of the permanent Curve of Temperature along a uniform thin Rod heated at one end. (Second Paper.)* By OLIVER J. LODGE, *D.Sc.*

THE first portion of a communication with the above title was printed in Part I. of the present volume of the Proceedings of the Society; but, a slip at the beginning having been noticed by Dr. Hopkinson which to some extent vitiated the results arrived at in the second part of the paper, this second half, consisting of sections 15–30, was suppressed, and not printed in the Proceedings of the Society, though it appeared in the Philosophical Magazine for April 1879. The present communication corrects and supplements the former paper, but, as it contains several references to the contents of the suppressed portion, this explanation is necessary.

The slip consisted in setting out with the ordinary equation to the curve of permanent temperature down a rod

$$\frac{d^2\theta}{dx^2}=\frac{Hp}{kq},$$

which is true when k is constant, and working with it as if it were equally valid when k is assumed to be variable. The oversight was inexcusable, because in § 2 of the paper referred to I indicated, for the sake of completeness, the ordinary way in which this fundamental equation is obtained, and I thoughtlessly wrote the gain of heat per second by an element of the rod at a distance x from the origin as

$$kqd\frac{d\theta}{dx}$$

as usual, instead of what it obviously becomes when k is not considered constant,

$$dkq\frac{d\theta}{dx}.$$

The term containing $\frac{dk}{d\theta}$ which I omitted is but a small one, however, and does not make very much difference to the result: hence the sections 16–20, though superseded by the present communication, are not exactly incorrect, but are first approximations; and the curve A spoken of in § 21, and drawn in Plate VII., does represent the character of the curve of temperature down a long iron rod *in vacuo*, with one end 300° hotter than the other. But no calculation of the variation coefficient of conductivity is likely to be possible by means of equations from which the term $\frac{dk}{d\theta}$ had been omitted.

30. With the exception of the correction now indicated in equations (1) and (3), the first fifteen sections of the paper are quite unaffected by the slip, and may remain as they stand, except that I have now a little more to say on the subject of §§ 5–9.

Professor Tait has been kind enough to send me a copy of his researches on "Thermal and Electric Conductivity," read before the Royal Society of Edinburgh in March and June 1878; and I find that he has given up his enticing speculation as to the inverse variation of thermometric conductivity with absolute temperature—and in fact that he believes iron to be possibly exceptional in the inverse connexion of conductivity and temperature, all other metals which he has subjected to experimental observation showing a slight *increase* of conduc-

tivity as the temperature rises. Prof. Tait's results are thus in opposition to the results of Prof. Angström for copper; but since Prof. Angström, in the interpretation of his very ingenious method of experiment, used the ordinary Fourier equations, formed on the supposition that k is constant, and that rate of cooling is proportional to excess of temperature, Prof. Tait does not consider his observations competent to decide a point as to the variability of k. Without venturing an opinion of my own on the subject, it is evident that this opposition is an additional reason for attacking the important question of the law of the variation of thermal conductivity with temperature.

Prof. Tait finds that a linear function of the temperature, $k=a+bt$, will express the value of the thermal conductivity according to his experimental results, at least in their present preliminary stage; and we saw in § 8 that Prof. Forbes's results for iron could be expressed nearly as

$$k=\cdot207(1-\cdot00144\,t);$$

hence instead of the law of variation of *thermometric* conductivity,

$$\frac{k}{c\rho}=\frac{A}{b+t},$$

assumed in the former paper, I shall now assume as the law of the variation of *thermal* conductivity,

$$k=b\kappa\left(1+\frac{1}{b}\,t\right)=\kappa(b+t),$$

where b is positive for metals whose conductivity increases with temperature, but both b and κ are negative for those whose conductivity diminishes with temperature, like iron.

It will be necessary to reckon temperature, as before, from the temperature of the enclosure ($v_0-274=t_0$) as zero; so instead of $b+t$ we write $m+\theta$, where $m=t_0+b$.

Similarly, for the law of variation of specific heat and density, we may assume (see § 8),

$$c\rho=\beta\sigma\left(1+\frac{1}{\beta}\,t\right)=\sigma(\beta+t)=\sigma(n+\theta),$$

where $n=t_0+\beta$ and is always positive, because the specific

heat of all metals except mercury appears to *increase* with temperature.

§§ 10–15 remain unaltered; but I shall only attempt the case of a rod *in vacuo*, as in air the integration appears impracticable. §§ 25–29 also remain practically unaltered.

31. We start, then, with the following equations to the variables, of which the numbering agrees with that of the corresponding equations in the preceding paper, whether they are the same or different; but the numbers are put into square brackets when for any reason the equation is modified. Numbers above 32 characterize those equations which did not appear in the former paper.

$$k=\kappa(m+\theta), \quad \text{[4]}$$

$$\therefore \frac{dk}{d\theta}=\kappa=\frac{k_0}{b}=\frac{k_0}{m+t_0}, \quad (33)$$

$$c\rho=\sigma(n+\theta), \quad (34)$$

$$\mathrm{H}=\frac{qc\rho}{p}\dot{\theta}, \quad (2)$$

$$\dot{\theta}=\mathrm{P}a^{v_0}(a^{\theta}-1)=\mathrm{R}(a^{\theta}-1). \quad (5')$$

And the fundamental equation is

$$dkq\frac{d\theta}{dx}=\mathrm{H}pdx; \quad \text{[1]}$$

or, neglecting the expansion by heat of the cross section q,

$$\frac{d}{dx}k\frac{d\theta}{dx}=\frac{\mathrm{H}p}{q}=c\rho\dot{\theta}. \quad \text{[3]}$$

This may be written

$$\frac{d^2\theta}{dx^2}+\frac{1}{k}\cdot\frac{dk}{d\theta}\left(\frac{d\theta}{dx}\right)^2=\frac{c\rho}{k}\dot{\theta},$$

whence, by help of [4] &c.,

$$\frac{d^2\theta}{dx^2}+\frac{1}{m+\theta}\left(\frac{d\theta}{dx}\right)^2=\frac{\sigma}{\kappa}\cdot\frac{n+\theta}{m+\theta}\cdot\dot{\theta}. \quad \text{[11]}$$

32. To integrate this once, make the substitution $\left(\frac{d\theta}{dx}\right)^2=z$, when [11] becomes

$$\frac{dz}{d\theta}+\frac{2z}{m+\theta}=2\frac{\sigma}{\kappa}\cdot\frac{n+\theta}{m+\theta}\cdot\dot{\theta}$$

which is of the form

$$\frac{dz}{d\theta} + Pz = Q,$$

whose integral is

$$z = e^{-\int P d\theta} \left\{ \int e^{\int P d\theta} . Q d\theta + C \right\}.$$

Hence the first integral of [11] is

$$\left(\frac{d\theta}{dx}\right)^2 = \frac{2\sigma}{\kappa(m+\theta)^2} \int_0^\theta (m+\theta)(n+\theta)\dot{\theta}\, d\theta,$$

the limits being taken to suit a rod whose length is unlimited (that is, one in which θ and $\frac{d\theta}{dx}$ vanish together)—a condition which is necessary for simplicity, as is explained in § 17.

We may now insert the value of $\dot{\theta}$ from (5) and write the above thus,

$$\frac{\kappa}{2R\sigma}\left\{(m+\theta)\frac{d\theta}{dx}\right\}^2 = \int_0^\theta (m+\theta)(n+\theta)(a^\theta - 1) d\theta$$

$$= \frac{a^\theta - 1}{\log a}\{M + (mn - M)\theta + \theta^2\} - M\theta, \quad . \quad [12]$$

where for shortness the letter M is written instead of the constant

$$mn - \frac{m+n}{\log a} + \frac{2}{(\log a)^2}.$$

33. So far we have proceeded with perfect accuracy; but in order to integrate the equation any further it is necessary to expand $a^\theta - 1$ and neglect higher powers of $\theta \log a$, as was done and justified for all probable values of θ in § 12. The approximation used is

$$a^\theta - 1 \doteq \theta \log a \{1 + \tfrac{1}{2}\theta \log a + \tfrac{1}{6}(\theta \log a)^2(1 + \tfrac{1}{8}\Theta \log a)\}. \quad (7')$$

Introducing this into [12], after writing γ for the small correction factor $\frac{1}{8}\Theta \log a$, and α for the perpetually occurring constant $\log a = \log_e 1{\cdot}0077 = \frac{1}{130}$, it becomes

$$\frac{\kappa}{2R\sigma}\left\{(m+\theta)\frac{d\theta}{dx}\right\}^2 =$$

$$= \theta\left(1 + \tfrac{1}{2}\alpha\theta + \frac{1+\gamma}{6}\alpha^2\theta^2\right)\left(M + (mn - M)\theta + \theta^2\right) - M\theta$$

$$= \tfrac{1}{2}\alpha\theta^2(A + B\theta + C\theta^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad [13]$$

where

$$\left.\begin{aligned} A&=\frac{1}{\alpha}\left(\alpha mn+m+n-\frac{2}{\alpha}\right),\\ B&=\frac{2-\gamma}{3}(m+n)+\frac{1+\gamma}{3}\left(\alpha mn+\frac{2}{\alpha}\right),\\ C&=1+\frac{1+\gamma}{3}\left(m+n-\frac{2}{\alpha}+\theta\right)\alpha. \end{aligned}\right\} \quad . \quad (35)$$

(Remember that $\frac{1}{\alpha}=130$, and that $\gamma=\frac{1}{8}\alpha\Theta=\frac{\Theta}{1000}$ practically.)

The coefficient C, therefore, is not quite constant, but depends upon θ. The dependence, however, is very slight, since $m+n$ is usually a large number, and α a small fraction; and it will be quite sufficient to write the average value $\frac{1}{2}\Theta$ instead of θ in the brackets of C, and thus to make it a constant. The usual relative sizes of the three constants are, A numerically much larger than B, and B numerically much larger than C.

For metals whose conductivity increases with temperature all three constants are positive; but for metals like iron, whose conductivity decreases with temperature and for which therefore m is negative, A and B are certain to be negative, while C is very likely to be positive but small. We can avoid this change of sign by noticing that when m is negative κ is also negative; hence, if we bring κ over to the right-hand side of [13], we shall get new constants, $\frac{A}{\kappa}$ and $\frac{B}{\kappa}$, which will be always positive, and $\frac{C}{\kappa}$, which is positive for all metals which have κ positive, but which is generally negative for those which, like iron, have κ negative. In all the following equations, wherever A and B appear alone, they may be always reckoned positive, because the κ has merely been cancelled out. In order that C may be negative, it is necessary not only that m shall be negative, but also that it shall be numerically greater than $n+\frac{1}{2}\Theta+\frac{1}{\alpha}$; hence this is the condition which will allow $\frac{C}{\kappa}$ to be positive when κ is negative.

34. It may be useful to calculate the numerical value of these constants for such metals as we at present possess any experimental data for. We will assume the temperature of the enclosure to be 0° C. (so that $m=b$ and $n=\beta$, see § 30),

and the highest observed point of temperature Θ on the rod to be 100° C.: then the following Table contains the values of the constants A, B, and C for iron and copper, together with certain ratios which will be used later. The row of numbers deduced from the experiments of Forbes, confirmed by Tait, are probably nearly accurate; the others are subject to discount.

Metal.	Value of β or n according to Bède and Fizeau.	Value of b or m according to		A.	B.	C.
Iron ...	760	Forbes ...	-700	$-578{,}000$	$-1{,}360$	$+\cdot577$
		Ångström	-640	$-504{,}600$	$-1{,}200$	$+\cdot746$
Copper..	2200	Ångström	-940	$-1{,}964{,}000$	$-5{,}063$	$+3\cdot4$
		Tait	$+2000$	$+4{,}912{,}200$	$+15{,}166$	$+12\cdot3$

$\frac{2A}{B}=r.$	$\frac{2C}{B}=s.$	$rs=\frac{4AC}{B^2}.$
$+850$	$-\cdot00085$	$-\cdot7225$
$+841$	$-\cdot00124$	$-1\cdot0428$
$+775\cdot7$	$-\cdot00134$	$-1\cdot0394$
$+647\cdot8$	$+\cdot00162$	$+1\cdot0494$

35. The integral of equation [13] may be written down without difficulty; and it constitutes the equation to the permanent curve of temperature down a long thin uniform cylindrical metal rod with a blackened surface heated at one end *in vacuo*.

$$\left[\log \frac{\left\{\frac{2A}{\theta}+B+2\sqrt{A}\sqrt{\left(\frac{A}{\theta^2}+\frac{B}{\theta}+C\right)}\right\}^{m\sqrt{\frac{\kappa}{A}}}}{\left\{2C\theta+B+2\sqrt{C}\sqrt{(A+B\theta+C\theta^2)}\right\}^{\sqrt{\frac{\kappa}{C}}}}\right]_{\Theta}^{\theta}=\sqrt{(R\sigma\alpha)}x. \quad [14]$$

For a much simpler and practically useful form of this equation see equation (40) § 38.

36. The expression on the left-hand side of this equation may be written in various ways; and its form is slightly different according as C is of the same or opposite sign to κ.

For the case when C and κ have the same sign (which according to Prof. Tait is probably most usual) we may write

it conveniently

$$\left[m\sqrt{\left(\frac{\kappa}{A}\right)}\sinh^{-1}\frac{\frac{2A}{\theta}+B}{\sqrt{(4AC-B^2)}}-\sqrt{\left(\frac{\kappa}{C}\right)}\sinh^{-1}\frac{2C\theta+B}{\sqrt{(4AC-B^2)}}\right]_\Theta^\theta=\sqrt{(R\sigma\alpha)}x. \quad . \quad . \quad (36)$$

But for the case when the signs of C and κ are opposite (as for iron), it becomes

$$\left[m\sqrt{\left(\frac{\kappa}{A}\right)}\cosh^{-1}\frac{\frac{2A}{\theta}+B}{\sqrt{(B^2-4AC)}}-\sqrt{\left(\frac{-\kappa}{C}\right)}\cos^{-1}\frac{2C\theta+B}{\sqrt{(B^2-4AC)}}\right]_\Theta^\theta=\sqrt{(R\sigma\alpha)}x. \quad . \quad . \quad (37)$$

Whether we write $\cos^{-1}$ or $\sin^{-1}$ in this equation only affects the sign of the term containing it. In equation (36), if 4AC is less than B^2 (which is unlikely), the terms in the denominators must be transposed and $\cosh^{-1}$ written for $\sinh^{-1}$.

Of the two terms in the brackets of these equations the first is by far the most important, and in the former paper is the only one which appeared (see equation 14); the second term has only a small effect on the curve, and this effect vanishes with C*. The occurrence of the inverse circular function in the curve of temperature along metals whose conductivity diminishes with temperature is peculiar; but C must always be very small for such metals; so that it does not make much difference. And this is a good thing, because when m is negative the θ occurring in the expression called C is of more relative importance, and C is therefore not so constant as when m is positive: but the approximation is always pretty good, the worst possible case being that of a supposititious metal with $m=-\left(n-\frac{2}{\alpha}\right)$, for which $C=1+\frac{\theta}{355}$.

37. We have thus obtained the equation to the curve of temperature expressing x as a function of θ. What we have now to do is to show how, from experimentally observed corresponding values of θ and x, the constants A, B, C can be determined, or at least such of them as are required for the determination of the constants κ and m. In order to deter-

* To avoid a possible misunderstanding, it may be well to say that this does not mean that the *term* vanishes, because of course it becomes infinite; but it means that the *effect* of the term vanishes, because when the limits are put in, the two things subtracted from one another are equal.

mine these constants we must apparently use some method of successive approximations; and the precise method adopted will probably be a matter of taste. I may, however, suggest the following as certainly applicable to the case of κ and m positive, *i. e.* to equation (36), and as inferentially applicable to the other case (37) if we can get some imaginary quantities to cancel each other. We will therefore proceed with the general case, and not trouble about whether the quantities are real or imaginary.

The quantity $\dfrac{\frac{2A}{\theta}+B}{\sqrt{(4AC-B^2)}}$ is always pretty large, even when θ has its maximum value Θ; call it y. Then writing $\sinh^{-1} y = \log(y+\sqrt{y^2+1})$, we see that, since y is large, y^2+1 is practically the same as y^2, and therefore that $\sinh^{-1} y \triangleq \log 2y$ to all intents and purposes. This approximation is always very close; and it is perfectly accurate when $rs=1$, *i. e.* when $4AC=B^2$. That this is not likely to be far from the case is illustrated in the Table (§ 34).

The quantity $\dfrac{2C\theta+B}{\sqrt{(4AC-B^2)}}$, however, is not large at all, but has a value not very different from unity. We had better therefore take its $\sinh^{-1}$ in the logarithmic form, and write (36) with the limits put in

$$m\sqrt{\frac{\kappa}{A}}\log\frac{\frac{2A}{\theta}+B}{\frac{2A}{\Theta}+B}$$
$$-\sqrt{\frac{\kappa}{C}}\log\frac{2C\theta+B+\sqrt{4AC+4BC\theta+4C^2\theta^2}}{2C\Theta+B+\sqrt{4AC+4BC\Theta+4C^2\Theta^2}}=\sqrt{(R\sigma\alpha)}x \quad . \quad . \quad . \quad (38)$$

or, writing $\frac{2A}{B}=r$ and $\frac{2C}{B}=s$,

$$\left[\frac{1+\frac{r}{\theta}}{1+\frac{r}{\Theta}}\right]\cdot\left[\frac{1+s\Theta+\sqrt{(rs+2s\Theta+s^2\Theta^2)}}{1+s\theta+\sqrt{(rs+2s\theta+s^2\theta^2)}}\right]^{\frac{1}{m}\sqrt{\frac{A}{C}}}=e^{\frac{1}{m}\sqrt{\left(\frac{AR\sigma\alpha}{\kappa}\right)}x}, \quad . \quad . \quad . \quad (39)$$

or

$$\left(1+\frac{r}{\theta}\right)\left(1+s\theta+\sqrt{rs+2s\theta+s^2\theta^2}\right)^{-\frac{1}{m}\sqrt{\frac{r}{s}}}=Ke^{\mu x}; \quad [19]$$

and the function of θ on the left-hand side of this equation increases in geometrical progression for an arithmetical increase of x.

38. So far we have been practically exact; but it is now necessary to introduce several approximations. Notice that r is a large constant and s a small one; so that $\frac{\theta}{r}$ and $s\theta$ are small quantities whose squares may be neglected for all probable values of θ. Moreover observe that $\frac{1}{m}\sqrt{\frac{r}{s}}$, though it may be imaginary, is never large and is often fractional. Write, therefore, the denominator of the left-hand side of [19] in the following successively approximate forms,

$$\left[1+s\theta+\sqrt{rs}\sqrt{\left(1+\frac{2\theta}{r}\right)}\right]^{\frac{1}{m}\sqrt{\frac{r}{s}}}$$

$$\doteq\left[1+s\theta+\sqrt{rs}\left(1+\frac{\theta}{r}\right)\right]^{\frac{1}{m}\sqrt{\frac{r}{s}}}$$

$$=\left[1+\sqrt{rs}+\theta\left(s+\sqrt{\frac{s}{r}}\right)\right]^{\frac{1}{m}\sqrt{\frac{r}{s}}}$$

$$\doteq(1+\sqrt{rs})^{\frac{1}{m}\sqrt{\frac{r}{s}}}+\frac{1}{m}\theta(\sqrt{rs}+1)(1+\sqrt{rs})^{\frac{1}{m}\sqrt{\left(\frac{r}{s}\right)}-1}$$

$$=\frac{m+\theta}{m}(1+\sqrt{rs})^{\frac{1}{m}\sqrt{\frac{r}{s}}}.$$

Hence we have as the quantity which is to go in geometrical progression in equation [19]

$$\frac{1+\frac{r}{\theta}}{m+\theta}=\frac{(1+\sqrt{rs})^{\frac{1}{m}\sqrt{\frac{r}{s}}}}{m}\,.\,\mathrm{K}\,e^{\mu x}, \quad . \quad [19']$$

and we may conveniently write equation [14] to the curve of temperature down the rod in the simple approximate form

$$\theta\,.\,\frac{m+\theta}{r+\theta}=\Theta\,.\,\frac{m+\Theta}{r+\Theta}\,e^{-\mu x}, \quad . \quad . \quad . \quad (40)$$

where it is to be remembered, see equation (39), that

$$\mu^2=\frac{\mathrm{AR}\sigma\alpha}{m^2\kappa}\doteq{}^{*}\frac{mn\sigma\mathrm{R}\alpha}{m^2\kappa}=\frac{c_0\rho_0}{k_0}\mathrm{R}\alpha=\frac{c_0\rho_0}{k_0}\,.\,\mathrm{P}a^{v_0}\log a, \qquad [21]$$

* The approximation in [21] consists in writing mn, the largest term of A, instead of A. It is unnecessary to work with this approximation; but it is useful as showing how naturally the various constants occur in μ.

and that

$$r=\frac{2A}{B}=\frac{6}{\alpha}\cdot\frac{\alpha mn+m+n-\frac{2}{\alpha}}{(2-\gamma)(m+n)+(1+\gamma)\left(\alpha mn+\frac{2}{\alpha}\right)}. \qquad (41)$$

39. There ought now to be no difficulty in calculating the constant m from observed corresponding values of θ and x. The following method has occurred to me; but there may be better ones. Let five temperatures, θ_0, θ_1, θ_2, θ_3, θ_4, be observed along the rod at distances from the origin x, $x+\xi$, $x+2\xi$, $x+3\xi$, and $x+4\xi$. Let the quantity which goes in geometrical progression, see equations (40) or [19′], be written

$$\frac{1+\frac{r}{\theta}}{1+\frac{\theta}{m}};$$

and for shortness call the numerator $\phi(\theta)$ and the denominator $\psi(\theta)$. Then of course

$$\frac{\phi_1}{\psi_1}\cdot\frac{\phi_3}{\psi_3}=\left(\frac{\phi_2}{\psi_2}\right)^2,$$

whence

$$\frac{\phi_1\phi_3-\phi_2^2}{\phi_2^2}=\frac{\psi_1\psi_3-\psi_2^2}{\psi_2^2}.$$

Similarly

$$\frac{\phi_0\phi_4-\phi_2^2}{\phi_2^2}=\frac{\psi_0\psi_4-\psi_2^2}{\psi_2^2}.$$

Therefore

$$\frac{\phi_1\phi_3-\phi_2^2}{\phi_0\phi_4-\phi_2^2}=\frac{\psi_1\psi_3-\psi_2^2}{\psi_0\psi_4-\psi_2^2},$$

which, being interpreted, is

$$\frac{\frac{1}{\theta_1}+\frac{1}{\theta_3}-\frac{2}{\theta_2}+r\left(\frac{1}{\theta_1\theta_3}-\frac{1}{\theta_2^2}\right)}{\frac{1}{\theta_0}+\frac{1}{\theta_4}-\frac{2}{\theta_2}+r\left(\frac{1}{\theta_0\theta_4}-\frac{1}{\theta_2^2}\right)}=\frac{(\theta_1+\theta_3-2\theta_2)m+\theta_1\theta_3-\theta_2^2}{(\theta_0+\theta_4-2\theta_2)m+\theta_0\theta_4-\theta_2^2}, \qquad (42)$$

the form of which we may abridge into

$$\frac{a_1+rb_1}{a_0+rb_0}=\frac{mc_1+d_1}{mc_0+d_0}$$

where the coefficients a, b, c, d are all known. This gives us

$$m=\frac{a_0d_1-a_1d_0+r(b_0d_1-b_1d_0)}{a_1c_0-a_0c_1+r(b_1c_0-b_0c_1)}. \quad . \quad . \quad . \quad (43)$$

40. Now looking at equation (41), we see that the value of r is $\frac{6}{\alpha}$, multiplied by a fraction which contains m indeed, but does not depend very much upon it; for, neglecting the small quantity γ, the fraction is

$$\frac{\alpha mn+m+n-\frac{2}{\alpha}}{\alpha mn+2(m+n)+\frac{2}{\alpha}};$$

and since m and n are both pretty large, the first term, which is the same in both numerator and denominator, is much the biggest, and accordingly the fraction is not very different from unity. It is likely to be greater than 1 when m is negative, and less than 1 when m is positive. Hence a first approximation to r is $\frac{6}{\alpha}$, or 780. The fact that r does not depend *much* upon m is apparent in the Table, § 34.

Making, then, a guess at r as 800 or so, we obtain from (43) a first approximation to m; and we can afterwards improve it by trial and error, so that the quantity $\theta\frac{m+\theta}{r+\theta}$ in equation (40) really does go in geometrical progression down the whole length of the rod—for instance, so that

$$\frac{1}{x}\log\frac{\Theta(m+\Theta)(r+\theta)}{\theta(m+\theta)(r+\Theta)}=\text{const}=\mu. \quad . \quad . \quad (44)$$

And then, having obtained the mean value of this constant μ, it is easy to calculate the absolute conductivity of the rod from [21] if one has determined R by experiments on cooling (see § 27). The *relative* thermometric conductivities of different metals at the temperature of the enclosure are simply inversely as μ^2 (see [21]).

41. The only experimental results already published which are even apparently at all suitable for applying the method to are those of Wiedemann and Franz; but it is impossible to get any results from these for reasons stated in § 28. It may, however, be interesting to see how far their numbers for an

iron rod in a vacuum will lend themselves to the equations now obtained, if m is assumed to be -700 and r to be 850, as in § 34.

The temperatures of the rod were read thermoelectrically, and stated in terms of galvanometer-deflections; and though a little Table is given showing what these deflections experimentally mean in Centigrade excess of temperature, the numbers actually used in their calculations are the deflections themselves, which are only roughly proportional to the temperature excesses. The following empirical relation between δ (the deflection) and θ (the excess of temperature) is deduced from their little comparison Table,

$$\theta = \tfrac{1}{4}\left(\delta + 2 - \frac{\delta}{10 - \frac{\delta}{100}}\right);$$

and this I have used to obtain the second column of the following Table:—

Iron (Wiedemann and Franz).

Galvanometer deflections. δ.	Excess of temperature in Centigrade degrees. θ.	$\log_{10}\left(\theta\frac{700-\theta}{850+\theta}\right) = \log f(\theta)$.	$\frac{1}{n-1}\log\frac{f(\theta_1)}{f(\theta_n)} = \mu\xi\log_{10}e$.	$\frac{f(\theta_{n-1})+f(\theta_{n+1})}{f(\theta_n)} = 2\cosh\mu\xi$.
230	50·8	1·5637		
153½	34·5	1·4142	·1495[a]	2·094[b]
100½	22·9	1·2487	·1575	2·168
67½	15·8	1·0965	·1557	2·068
42	10·1	·9086	·1638	2·172
25·7	6·3	·7079	·1712	2·147
13·2	3·5	·4558	·1846	

[a] The mean of the first four numbers in this column is ·1566; therefore

$$\mu\xi = \cdot 3606.$$

[b] The mean of these numbers is 2·130; therefore

$$\mu\xi \simeq \sqrt{(2\cosh\mu\xi - 2)} = \cdot 3606.$$

The third column contains the logarithm of the quantity which ought to go in geometrical progression. The fourth column shows that this law of progress is moderately true for the first four numbers, but that the numbers which ought to be constant exhibit a decided increase towards the cooler

end of the bar, probably because the bar was so short that the flow of heat along it extended through the point where $\theta=0$—which is contrary to our hypothesis (§§ 32 and 17) that $\frac{d\theta}{dx}$ shall vanish with θ. The distance ξ was 2·6 centimetres.

The next Table shows their results for silver tabulated in the same way, and on the assumption that $m=+1000$.

Silver (Wiedemann and Franz).

δ.	θ.	$\log\left(\theta\frac{1000+\theta}{700+\theta}\right)$ $=\log f(\theta)$.	$\frac{\log f(\theta_1)-\log f(\theta^n)}{n-1}$ $=\mu\xi\log e$.	$\frac{f(\theta_{n-1})+f(\theta_{n+1})}{f(\theta^n)}$ $=2\cosh\mu\xi$.
194	42·5	1·7758		
167	37·3	1·7200	·0558[a]	1·977[b]
142	32·0	1·6544	·0607	2·024
122	27·5	1·5893	·0621	2·018
104	23·5	1·5217	·0635	2·030
88½	20·2	1·4568	·0638	2·036
75½	17·7	1·3994	·0627	

[a] The mean of these numbers is ·0614; hence

$$\mu\xi=\frac{\cdot 0614}{\cdot 4343}=\cdot 141.$$

[b] The mean of these is 2·017; hence

$$\mu\xi=\sqrt{\cdot 017}=\cdot 1304.$$

42. Although the whole investigation applies only to the case of a long rod, yet it seems extremely possible that something very like the proper equation to the curve of temperature down a short rod with given temperature at its two ends can be written down from equation (40) by the addition of another term to the right-hand side, thus,

$$\theta\frac{m+\theta}{r+\theta}=\mathrm{A}e^{\mu x}+\mathrm{B}e^{-\mu x}, \quad . \quad . \quad . \quad . \quad (45)$$

just as the ordinary equation for an infinite rod

$$\theta=\Theta e^{-\mu x}$$

becomes

$$\theta=\mathrm{A}e^{\mu x}+\mathrm{B}e^{-\mu x}$$

for a short one. If so, the conductivity-constant μ and the

variation-constant m would be best determined from the relation

$$\frac{\theta_1 \dfrac{m+\theta_1}{r+\theta_1} + \theta_3 \dfrac{m+\theta_3}{r+\theta_3}}{\theta_2 \dfrac{m+\theta_2}{r+\theta_2}} = \text{const} = 2 \cosh \mu\xi \quad . \quad . \quad [25]$$

by some such process as is given in §§ 23–25.

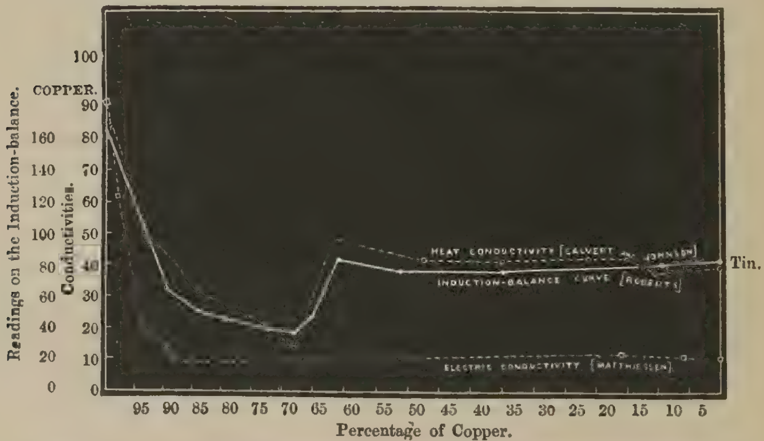
The extent of the constancy of this quotient with Wiedemann's numbers is therefore exhibited in the last column of the preceding tables; and the value of $\mu\xi$ is obtained from this column in order to compare it with that obtained from the column preceding. In the case of iron the two values accidentally agree exactly.

43. As regards the concluding section of the former paper (the suggestions for future experiment), I have seen no reason to modify it in any way. I still think the rods should be examined by a thermoelectric compensation method (*i. e.* with the use of a bridge wire and slider, and with fixed thermoelectric joints on the rod) after their surfaces have been uniformly coated with stove black-lead. The exhaustion of the chamber containing the rod should proceed to the "neutral-point" observed by Mr. Crookes, and employed by Mr. Poynting in his delicate weighing experiment (Proc. Roy. Soc.), "when convection currents have ceased and the radiometer effect has not yet begun." It is probable that at such a point (if it really exists, as I suppose it does) Dulong and Petit's law of cooling, assumed all through this paper, would hold with perfect accuracy; and I think that with very careful observations of temperature over a sufficient range on long thin rods, there would be no insuperable obstacle to obtaining rapidly the absolute conductivity and the coefficient of its variation with temperature of as many metals and alloys as one pleased.

I have taken considerable care; and I hope that all the calculations in the present paper are free from error; but they are rather long and unsymmetrical and therefore tiresome, and I have not ventured to ask any one to read over the proof for this very reason.

XXII. *On an Analogy between the Conductivity for Heat and the Induction-Balance Effect of Copper-Tin Alloys.* By W. Chandler Roberts, *F.R.S.**

In a paper submitted to the Society in June last, I pointed out that the results obtained when copper-tin alloys are examined by the aid of the induction-balance does not correspond with Matthiessen's determinations of the electric conductivities of the same alloys. The nature of the induction-curve remained therefore more or less obscure; but I have recently observed a remarkable resemblance between my own results † and those given by Calvert and Johnson ‡ for the conductivity of heat, which had hitherto escaped me, as the authors did not plot their figures. The following diagram shows the relation of the curve published by me in July last with those of Calvert and Johnson and Matthiessen respectively, which I find for the first time placed in juxtaposition in a valuable report on the copper-tin alloys recently issued by the United-States Government§.



Note.—The curves for the conductivity of heat and electricity are taken from Plate xiv. of the Report on Copper-Tin Alloys, above referred to. The latter curve should, however, be moved slightly to the left, as Matthiessen's numbers represent *volumes*, and not weights, per cent. of copper.

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* Read November 8th.

† Phil. Mag. (5) vol. viii. p. 57, 1879.

‡ Phil. Trans. vol. cxlviii. 1858, p. 349.

§ Report made under the Direction of the Committee on Metallic Alloys. Washington, 1879.

The earlier results, which were obtained by observations on blocks of alloy 18 millims. square and 7 millims. thick, have since been verified (except in the case of pure copper, which as originally given is too low) by a series of disks 24 millims. in diameter and 3 millims. thick, formed in the lathe, or, in the case of the very brittle ones, by filing.

The close agreement between the conductivity for heat and electricity has, as is well known, been pointed out by Wiedemann and Franz *; but as the similarity has hitherto, I believe, only been observed in isolated cases, the above results on an extended series of alloys may be of interest.

Calvert and Johnson claim a high degree of accuracy for their results; but they were subjected to adverse criticism, as Wiedemann † objected to the apparatus they adopted and to the small size of the bars employed. It is therefore the more important to point to the almost absolute identity of the two curves representing phenomena of which the manifestations are so different.

The respective values of the alloys were ascertained by placing the metal to be examined on one side of the balance and by superposing a graduated wedge-shaped scale of zinc over the opposing coil, as has already been described by Prof. Hughes ‡. The figures given, can however, only be considered to represent the general relation among themselves of the various alloys in the series, as the zinc scale does not give absolute values.

The induction-balance curve, as I have previously pointed out, bears an evident relation to the curve representing the density of the same alloys; but its divergence from Matthiessen's curve § of conductivity is singular. This may perhaps be explained by the fact that Matthiessen does not appear to have examined any alloy between those which contain respectively 16·4 and 85·1 volumes per cent. of copper, probably because the alloys between these points are too brittle to permit their being formed into wire. It may be noted, however, that the curve of the tin-gold alloys, which belong to

* Poggendorff's *Annalen*, vol. lxxxix. [1853], pp. 497–531.

† *Ibid.* vol. cviii. [1859], pp. 393–407.

‡ Phil. Mag. (5) vol. viii. p. 50.

§ Brit. Assoc. Report, 1863, p. 37, and Chem. Soc. Journ. 1867, p. 212.

the same group as the copper-tin, is given in his well-known paper * in 1860, and does not show an unbroken line in the horizontal portion. Certain intermediate alloys, in the form of rods, are now being examined by Dr. Lodge, whose results will doubtless clear up the point.

The alloys which occupy the critical points of the induction-balance curve are very interesting. They may be represented by the formulæ $SnCu_3$ and $SnCu_4$ respectively; and although they only vary by a single equivalent, or by 6·49 per cent. of copper, their structure and appearance differ widely. The latter, $SnCu_4$, is a speculum-metal. It has a large conchoidal fracture and a yellow-grey tint. $SnCu_3$, on the other hand, has a blue-grey colour and a coarse surface of interrupted crystalline planes. By successive additions of copper, this alloy seems to pass into the other without any sharp break.

Possibly both alloys are chemical combinations; and the difference of their ordinates probably marks a different allotropic state. For further information on such questions, however, we may look with confidence to Prof. Hughes's beautiful and simple instrument, which will also help us to determine whether the relation between conductivity for heat and electricity is really as exact as it has hitherto been supposed to be.

XXIII. *Note on a Determination of the Specific Electrical Resistance of certain Copper-Tin Alloys.* By Oliver J. Lodge, *D.Sc., Assistant Professor of Physics in University College, London.*

On the 5th of November I received from Mr. Chandler Roberts three rods of certain very brittle copper-tin alloys which he had with some difficulty cast, in order that I might determine their specific resistance per unit of volume by ordinary processes, because the readings which the induction-balance gave for disks of these three metals indicated that their conductivities did not by any means agree with Matthiessen's curve of electric conductivity for copper-tin alloys, though, on the other hand, they did agree with some severely

* Phil. Trans. vol. cl. p. 161.

criticised and discarded experiments of Messrs. Calvert and Johnson on the conductivities of these same alloys for *heat*.

The rods, which are hereafter spoken of as A, B, and C, were about 9 millimetres thick and from 30 to 40 centimetres long. The method which I employed to determine their conductivity was scarcely altered from that which was described by Prof. Foster and myself in this Journal in 1875*; and the results which I have obtained I believe to be somewhat accurate. It was not worth while to aim at any very *excessive* accuracy, though the method is quite capable of it (by a calibration of the bridge-wire), because the temperature of the alloys was a little indefinite, being simply that of the room.

The three rods were laid end to end in a groove in a long piece of wood so that they just projected above its surface; and their ends were then screwed up into secure contact with pads of tinfoil between. The resistance of the contacts is not of the slightest consequence, provided it remains constant and is moderately small, all measurements being made within the length of the rod itself.

A bridge-slider was arranged so as to make contact with one of the rods at a point whose position could be read by a millimetre-scale fixed at the proper level. The length of the portion of the rod examined could be ascertained within the tenth of a millimetre; and the resistance of this portion could also be read with great accuracy. The least satisfactory part of the measurement is the gauging of the diameter of the rod; for they are not perfectly cylindrical, and the *average* cross section has to be taken. I would have had the rods turned; only two of them, viz. B and C, are far too brittle to be touched with any tool.

The observation of resistance consisted in placing the slider at a definite point on one of the rods, and feeling about on the bridge-wire with another slider for a point of the same potential. The slider on the rod was then shifted to another part of the same rod and the bridge-slider shifted correspondingly. The shifts on the bridge-wire (German silver) were of course rather small; but they could be read to the tenth of a millimetre; and, moreover, the resistances in the circuit were so adjusted that any step along the rods corresponded to a step of 3·62 times

* Foster and Lodge, "On the Flow of Electricity in a Plane.—Part II.," Phil. Mag. December 1875.

the resistance along the bridge-wire. It is needless to say that a weak battery and a reflecting galvanometer were used, and that the contacts were only made momentarily. As an example of the error to be expected I may quote the following.

Two different steps along rod A gave, as the step on the bridge-wire corresponding to the resistance of 1 centimetre length of the rod A, the numbers

·0328 and ·0325 centim.

Two different steps along rod B gave

·0269 and ·0270 centim.

Different steps along rod C, which was a trifle less uniform, gave

·123, ·121, ·1177, ·1173, ·1185, ·1171 centim.

As the step ·1185 was deduced from the entire length of the rod, I have taken this as the mean. The steps which gave ·123 and ·121 were short ones; and their probable error is therefore greater than that of the others. Moreover they were made at the thinner end of the not quite cylindrical rod. As regards the mean cross section of the rods, it was estimated by taking a number of readings of the diameters in all parts of the rods by means of a micrometer-gauge reading to the hundredth of a millimetre.

The mean diameter of rod A was ·879 centim.
" " " B " ·873 "
" " " C " ·869 "

The rods had been cast in the same mould; but C had contracted more than the others.

On the 7th of November Mr. Roberts sent me three more rods, which I labelled D, E, F, and then determined their resistance in the same way.

The mean diameter of these rods, which were perfectly strong and might have been drawn into wire, was

D . . . . . . ·880 centim.
E . . . . . . ·862 "
F . . . . . . ·872 "

These had had their surfaces filed after casting.

To get the value of the bridge-wire-readings in absolute measure, a half-ohm coil of German-silver wire at the temperature of the room (*i. e.* of the bridge-wire) was inserted in the place of the rods, and readings taken of the steps corresponding to it. The scale of magnification was of course again observed. Instead of being about 3·6, it was now about $\frac{1}{3}$.

The result of the whole series of measurements may be summed up in the following Table:—

Specific Resistances of the six Copper-Tin Alloys in square centimetres per second.

A	12960
B	10960
C	47660
D	11830
E	17090
F	15270

I consider these accurate to the second figure, the third figure is probably affected by experimental errors.

If we choose to express the results in terms of conductivity, we have the following:—

Conductivity of a Centimetre Cube of the six Alloys in B.A. units.

A	77100
B	91200
C	21000
D	84500
E	58500
F	65500

The specific resistance of the alloy C is thus four times that of B or D.

The composition of the alloys by weight, Mr. Roberts informs me, is as follows:—

A	19·2 per cent. of copper	80·8 tin.
B	61·8 „ „	38·2 „
C	68·3 „ „	31·7 „
D	0·0 „ „	100·0 „
E	87·4 „ „	12·6 „
F	90·3 „ „	9·7 „

Some of these alloys were examined by Matthiessen, viz. A, D, E, and F; and the conductivity of a wire 1 metre long and 1 millimetre thick is stated by him to be, in terms of the B.A. unit,

of the metal A	6·0
„ D (pure tin) . .	5·9
„ E	4·2
„ F	5·9

To reduce these numbers from a wire to a centimetre cube we must multiply them by $\frac{40000}{\pi}$, when they become

for A	. . . . . .	76400
„ D	. . . . . .	75100
„ E	. . . . . .	53500
„ F	. . . . . .	75100

of which, therefore, A and E agree tolerably with my measurements on the rods.

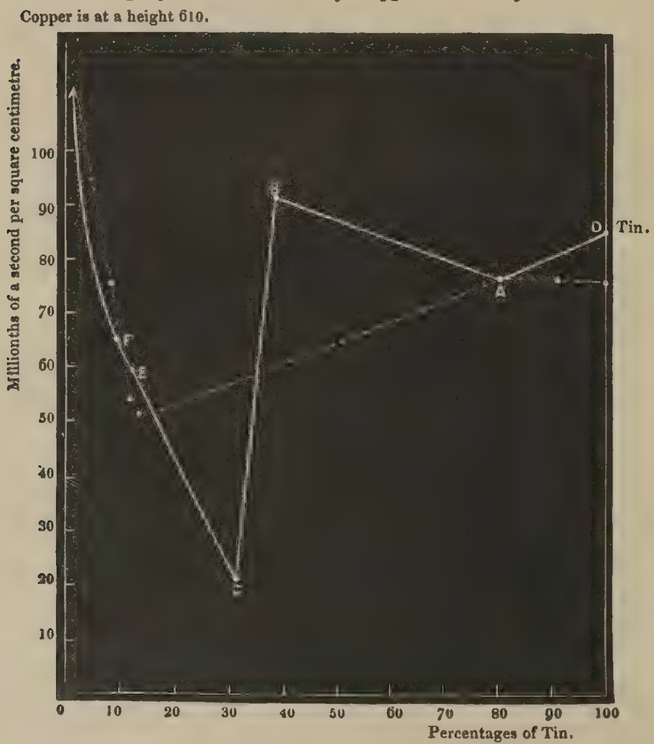
Matthiessen also measured the conductivity of pure copper, his number when reduced being

Copper 610000

or about eight times that of pure tin.

The results may be plotted in order to compare them more readily with the readings given by the induction-balance. The abscissæ represent percentages of tin; the ordinates represent conductivities divided by 1000.

Specific Conductivities of Copper-Tin Alloys.



The continuous broken line joins together my six points of observation. The dotted line is Matthiessen's curve of conductivity on the same scale, viz. absolute measure, with some of his points of observation marked on it; but no alloys between somewhere about E and A seem to have been actually examined by him.

The position of pure copper (from Matthiessen's measurement) cannot be shown in the diagram: the conductivity decreases at such a tremendous rate with the slightest percentage of tin that the ordinate for pure copper is 600 scale-divisions, whereas the greatest height shown in the diagram, that of the point B, is only about 90.

It is unnecessary to call attention to the extraordinary behaviour of the alloy C containing 31·7 per cent. of tin (that is, the alloy $SnCu_4$), or to point out how easily it might have been supposed to be a distinct metal.

XXIV. *On a Suggestion as to the Constitution of Chlorine, offered by the Dynamical Theory of Gases. By* A. W. RÜCKER, *M.A., Professor of Physics in the Yorkshire College, Leeds**.

IF a gas of density δ consists of molecules each of which possesses m degrees of freedom, and if also the intermolecular forces are negligible, the specific heats at constant pressure (c_p) and at constant volume (c_v) are connected by the two well-known equations

$$(c_p - c_v)\delta = \cdot 0694, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{c_p}{c_v} = 1 + \frac{2}{m+e}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where e is a quantity which depends upon the potential energy of a molecule. Hence, if c_p is given by experiment, c_v can be calculated from the first of these equations; and then $m+e$ is known from the second.

The accuracy of the value of $m+e$ thus deduced will depend upon that of c_p, and on the legitimacy of the application of the two equations to the gas or vapour under consideration.

* Read November 22nd.

With respect to the first of these points, it may be remarked that E. Wiedemann has recently (Pogg. *Ann.* Bd. clvii. p. 1, 1876, and Wied. *Ann.* Bd. ii. p. 195, 1877) determined the specific heats at constant pressure of 14 out of the 35 gases and vapours studied by Regnault. The difference between the results of the two investigators amounts in two cases only (ethylene and ammonia) to 6 per cent.; in three cases it is about 5 per cent., and in all the others less. Thus even on the assumption that the later experiments are absolutely correct, it follows that Regnault's numbers may be trusted to 6 per cent. His results, however, can only be taken as true for the particular temperatures at which the experiments were made, as Wiedemann shows that in all but the most perfect gases the specific heat at constant pressure varies considerably with the temperature.

More recently still, Wüllner (Wied. *Ann.* Bd. iv. p. 321, 1878), using Kundt and Warburg's method, has determined the ratio of the specific heats of air, carbonic oxide, carbonic acid, nitrous acid, ethylene, and ammonia at 0° and 100° C. He finds that for gases which obey Boyle's law, c_p and c_v are constant, but that in the case of less perfectly gaseous bodies they increase with the temperature. The difference between them, however, is always very approximately constant and equal to the theoretical number—thus justifying the application of the first of the above equations to imperfect gases, and proving that the observed increase in the specific heats is due to work done within the molecules, and not against the intermolecular forces, which must therefore be negligible.

On the whole, then, the result of these researches is to show that $m+e$ can be calculated very approximately from the above equations if c_p is given, and that Regnault's values of this quantity are probably trustworthy to 6 per cent.

One of the chief difficulties of the thermodynamic theory of gases has been to attribute to m and e values which would at once lead to the observed ratios of c_p to c_v, and satisfy any rational supposition as to the interior mechanism of a molecule. Kundt and Warburg proved that for mercury vapour $\frac{c_p}{c_v}=1{\cdot}666$, which is consistent only with the supposition that the atoms of that substance are smooth rigid spheres. Boltz-

mann (Pogg. *Ann.* Bd. clx. p.175, 1877) and Bosanquet (Phil. Mag. April 1877) have since drawn attention to the fact that, for a smooth rigid surface of revolution, $m=5$ and $e=0$, which would make $\frac{c_p}{c_v}=1{\cdot}4$. The fact, therefore, that this number agrees very closely with those given by experiment for a large number of gases (air, O, N, H, CO, and NO) would be accounted for by supposing their molecules to be surfaces of revolution. This condition would be fulfilled, as is pointed out by Mr. Bosanquet, by two spheres rigidly united, and would thus accord well with our conception of the atomic constitution of the above gases. It would perhaps be better to regard the spheres not as rigidly united, but as bound together by forces which prevent the separation of their surfaces, while leaving them otherwise free to move. The required five degrees of freedom would thus be obtained, and the hypothesis would better coincide with the supposition of the union of two *smooth* spheres to form the ultimate particles of the gases.

For a discussion of the difficulties offered to any such theory by the spectroscope I must refer to the above papers; my present purpose is to point out an interesting fact connected with its application to chlorine and its compounds.

The maximum number of degrees of freedom which a molecule composed of n smooth rigid spheres could possess would be $3n$; but the forces in play between the spheres might, as in the case of air and the other gases above referred to, reduce the number to much below this amount. Thus the value of $m+e$ could not exceed, but might be less than $3n+e$. For gases in which the molecule consists of two atoms, $e=0$; but in the cases of more complex combinations we can say only that, *cæteris paribus*, we should expect that its value would increase with the number of atoms in the molecule. Bearing these facts in mind, the following Tables lead to a curious result. In the first column of each are placed the symbols of the substances referred to. In the second columns are the ratios of the specific heats, deduced (except in the case of mercury) from Regnault's experiments by the use of equation (1). In the third are the values of $m+e$ (omitting fractions) deduced by equation (2). In the fourth are the values of $3n$, where n is the number of atoms of which the molecule is composed.

TABLE I.

I.	II.	III.	IV.
	$\frac{c_p}{c_v}$.	$m+e$.	$3n$.
Hg	1·666	3	3
Air	1·413	5	6
O_2	1·403	5	6
N_2	1·409	5	6
H_2	1·417	5	6
NO	1·403	5	6
CO	1·416	5	6
N_2O	1·243	8	9
H_2O	1·302	7	9
H_2S	1·335	6	9
CO_2	1·265	8	9
SO_2	1·276	7	9
CS_2	1·198	10	9
NH_3	1·300	7	12
CH_4	1·290	7	15
C_2H_4	1·144	14	18
C_2H_6O	1·110	18	27
$C_4H_{10}O$......	1·059	34	45
$C_4H_{10}S$......	1·059	34	45
C_3H_5N	1·283	7	27
$C_4H_8O_2$......	1·060	33	42
C_3H_6O	1·091	22	30
C_6H_6	1·072	28	36
$C_{10}H_{16}$	1·031	64	78

TABLE II.

I.	II.	III.	IV.	V.
	$\frac{c_p}{c_v}$.	$m+e$.	$3n$.	
Cl_2	1·286	7	6	18
HCl	1·408	5	6	12
C_2H_5Cl......	1·131	15	24	30
$CHCl_3$	1·118	17	15	33
$C_2H_4Cl_2$ ...	1·097	21	24	36
PCl_3	1·122	16	12	30
$AsCl_3$	1·111	18	12	30
$SiCl_4$	1·098	20	15	39
$SnCl_4$	1·092	22	15	39
$TiCl_4$	1·089	22	15	39

The first Table contains a number of simple and more or less complex compound gases and vapours; the second is confined to chlorine and its compounds alone. The difference between the two is most marked. In the first the value of $m+e$ is for every substance (with one exception) less than $3n$,

or than the maximum possible value of m. In the second the reverse statement holds good in more than two thirds of the whole number of cases.

This difference can hardly be accidental; nor can it be explained by an error of 6 per cent. in Regnault's experiments. It might be accounted for by supposing that in the case of chlorine e is abnormally large—and that this gas differs from others in which the molecule is built up of two spheres, in that the spheres are not necessarily in contact, and are probably therefore less firmly united.

Another supposition, however, would meet the case equally well, viz. that n has been taken too small, that the symbol Cl_2 is incorrect, and that the atoms of chlorine, and therefore the molecules of its compounds, contain a larger number of subatoms or atoms than has been supposed. It need hardly be added that this supposition fits in most satisfactorily with the results of the recent researches of Prof. Victor Meyer on the vapour-density of chlorine; and in the fifth column of Table II. are given the values of $3n$ calculated on the assumption that throughout the first column we ought to write Cl_3 for Cl. Hydrochloric acid now offers a difficulty, as a body composed of four spheres could only possess so small a number of degrees of freedom as five if the spheres were rigidly connected with their centres in one straight line. With this exception, however, columns III. and V. of Table II. now present differences of the same sign and order as those in the corresponding columns in Table I.

The number of degrees of freedom attributed to each substance may perhaps be wrong by one in some of the simpler bodies, and by rather larger numbers in some of the more complex; but the general character of the Tables is probably beyond the reach of any such changes. An error *e. g.* of 6 per cent. in the specific heat at constant pressure of the tetrachlorides would only reduce the number of their degrees of freedom by one. Much greater alterations would be introduced by taking the specific heats at other temperatures than those at which they were determined by Regnault. Chloroform is the only compound of chlorine of which the law of the variation of the specific heat with the temperature was studied by Wiedemann; and using his results, I find that at 0° C. the

number of degrees of freedom of this substance would be one less instead of two greater than 15.

In spite, however, of the uncertainty thus introduced, the comparison of the two Tables is sufficiently suggestive to induce me to lay them before the Physical Society. The fact that the application of the theory of gases to the specific heats of a large number of substances, determined as far as might be under similar circumstances, leads to the alternatives that the atoms of which a molecule of free chlorine is composed are either less strongly united or are more numerous than in the case of other elements, is not unimportant.

It remains to add that the ratios of the specific heats of bromine and of the only one of its compounds which has been studied agree with those of chlorine and the corresponding chlorine compound.

TABLE III.

I.	II.	III.	IV.	V.
	$\frac{c_p}{c_v}$.	$m+e$.	$3n$.	
Br_2	1·302	7	6	18
C_2H_5Br ...	1·114	18	24	30

Note.—Since the above was written, the conclusions at first drawn from Prof. V. Meyer's research have been questioned (Chem. News, Nov. 21, 1879, p. 244) on the ground that experiments made by Seelheim, of Utrecht, indicate the possibility of the formation of a volatile chloride of platinum at high temperatures. However this may be, the cause of the anomalous specific heats of chlorine and its compounds remains to be explained; and the above statement of the alternative suppositions to which the theory discussed leads may not be uninteresting, even if it should be proved that one of them lacks the support which, at the time of writing, Prof. V. Meyer's research was supposed to afford it.

XXV. *On the Graduation of the Sonometer.* By J. H. Poynting, *M.A., Fellow of Trinity College, Cambridge**.

It seems likely that such valuable results will be obtained by means of Professor Hughes's sonometer, that it is desirable that some method should be employed to turn its at present arbitrary readings into absolute measure, so that, for instance, the induced currents caused by different metals in the induction-balance may be measured and compared with each other.

In Maxwell's 'Electricity,' vol. ii. chap. xiv., the general formula is given for the coefficient of induction of one circular circuit on another. Adapting this to the case where two equal circular circuits are on the same axis at a distance apart greater than the radius of the coils, the following formula is obtained.

Let $a =$ distance between centres,
$b =$ radius of either circle,
$c =$ distance of either circumference from centre of other,
M = coefficient of induction.

Then

$$M = -\frac{4\pi^2 b^4}{c^3}\left\{\frac{1}{2} - \frac{3}{4}\frac{b^2}{c^2} + \frac{15}{8}\frac{b^4}{c^4} - \frac{35}{8}\frac{b^6}{c^6} + \frac{2835}{256}\frac{b^8}{c^8} - \&c.\right. \quad (1)$$

or

$$= -\frac{4\pi^2 b^4}{a^3}\left\{\frac{1}{2} - \frac{3}{2}\frac{b^2}{a^2} + \frac{75}{16}\frac{b^4}{a^4} - \frac{490}{32}\frac{b^6}{a^6} + \frac{24570}{256}\frac{b^8}{a^8} - \&c.\right. \quad (2)$$

Of these the latter uses directly the distance between the centres, the observed quantity—but is not nearly so convergent as the former, in which c may be at once deduced from $c = \sqrt{a^2 + b^2}$.

To obtain formulæ which might be strictly applied to the sonometer, we should have to consider the more general case of two coils of unequal radii b and β, for which I have found the formula corresponding to (2), viz.

$$M = \frac{4\pi b^2\beta^2}{a^3}\left(\frac{1}{2} - \frac{3}{4}\frac{b^2+\beta^2}{a^2} + \frac{15}{16}\frac{b^4+3b^2\beta^2+\beta^4}{a^4}\right.$$
$$\left. - \frac{35}{32}\frac{b^6+6b^4\beta^2+6b^2\beta^4+\beta^6}{a^6} + \&c. \right. \quad (3)$$

* Read December 13th, 1879.

We should then have to take the finite integrals of each term between the limiting values of b and β. But this would be exceedingly complicated and would require a knowledge of all the details of construction; and we may at least get a first approximation to the true result by replacing the coils by a single one of a radius intermediate between the greatest and least radii.

In Prof. Hughes's paper (Phil. Mag. July 1879) he gives the internal and external radii of his coils as 15 millims. and 27·5 millims. respectively. I have considered, then, that 25 millims. will give results not very far from the truth; and as it makes the calculations considerably easier, I have taken that as the value of b and applied the formulæ to the numbers given in the paper. The resultant current in the middle coil was zero when it was distant 47 millims. from one end and 200 from the other. This enables us to find the ratio between the number of turns in the two ends at least sufficiently nearly to apply to some of the results.

Let M_1 be the coefficient of induction of the larger coil on the movable one, M_2 that of the smaller, the former having m turns, the latter n. When the movable coil was 200 millims. from the large and 47 millims. from the small coil, since there was no induced current,

$$mM_1 = nM_2.$$

Applying formula (1), we have c for the larger coil

$$=\sqrt{200^2+25^2}=201\cdot5,$$

and for the smaller coil

$$c=\sqrt{47^2+25^2}=53\cdot2,$$

b being the same for both. Then

$$\frac{m}{(201\cdot5)^3}\left\{\frac{1}{2}-\frac{3}{4}\left(\frac{25}{201\cdot5}\right)^2+\frac{15}{8}\left(\frac{25}{201\cdot5}\right)^4-\&c.\right\}$$

$$=\frac{n}{(53\cdot2)^3}\left\{\frac{1}{2}-\frac{3}{4}\left(\frac{25}{53\cdot2}\right)^2+\frac{15}{8}\left(\frac{25}{53\cdot2}\right)^4-\frac{35}{8}\left(\frac{25}{53\cdot2}\right)^6\right.$$

$$\left.+\frac{2835}{256}\left(\frac{25}{53\cdot2}\right)^8-\&c.\right.$$

Multiplying each side by 2 and finding the successive terms,

$$m \times \frac{122}{10^9}\{1 - \cdot 02308 + \cdot 00088 - \&c.\}$$

$$= n \times \frac{6645}{10^9}\{1 - \cdot 33123 + \cdot 18286 - \cdot 09422 + \cdot 02633,$$

or

$$\frac{m}{n} = 43 \cdot 6.$$

I have applied the formula to the results for various metals given by Prof. Hughes in a table in his paper. In the table below, in the first column are Prof. Hughes's numbers, *i. e.* distances from the point of no induction. In the second are numbers proportional to $mM_1 - nM_2$; where M_1, M_2 are the coefficients of induction of two simple coils calculated on the above hypothesis, m and n the number of turns in the two respectively. In the third column are the resistances for bars of the metal 100 millims. long and 1 millim. in diameter (Jenkin, p. 249). In the last column are the products of the numbers in the two preceding columns.

Metal.	Distance from point of no induction.	$mM_1 - nM_2$, proportional to	R.	$(mM_1 - nM_2)$R.
Silver	125	178	·21	37·4
Gold	117	135	·27	36·5
Aluminium...	112	116	·375	43·5
Copper	100	84	·21	17·6
Zinc	80	50·1	·72	36·1
Tin...............	74	44·6	1·70	75·8
Iron	45	22·46	1·25	28·1
Lead	38	18·87	2·5	47·2
Antimony	35	17·35	4·5	78·1
Bismuth	10	5·75	16·8	96·6

Mercury has been omitted, as it gives a very much higher value than any of the others. Were the induced currents in the induction-balance proportional to the resistances given in the table, the numbers in the last column would of course be all the same. The deviations from equality are far greater than could be accounted for by errors in the approximations I have adopted, especially for the metals not at the beginning or end of the list. Hence we are driven to conclude, either that the resistances of the metals given in the tables are not

the same as the resistances of the metals used by Prof. Hughes, or that the induced current is not proportional to the conductivity of the metal.

It should be noticed that the method of measuring currents by the sonometer assumes that the telephone integrates, as it were, the current; *i. e.* the loudness of the sound depends only on the total current, not on the time during which the current is passing, provided that the time be very short. I do not know whether this point has been investigated; but if not, it would probably be easy to examine it by means of the sonometer. It would be advisable to modify the instrument in such a way that the formulæ might be more easily employed, and that the approximations might be nearer to the truth.

The formulæ used in this paper may be obtained as follows, the method being adapted from that given in Maxwell.

The potential of a circular unit current at any point is the same as that of a magnetic shell of unit strength bounded by the circuit. This, again, is the same as the attraction of a thin plate of matter of unit surface-density in a direction perpendicular to the plane of the plate. If ω be the attraction of a plate of radius b, at a point distant b from the plate along its axis,

$$\omega = 2\pi\left(1 - \frac{1}{\sqrt{1+\frac{b^2}{c^3}}}\right)$$

$$= 2\pi\left\{\frac{1}{2}\frac{b^2}{c^2} - \frac{1.3}{2.4}\frac{b}{c^4} + \frac{1.3.5}{2.4.6}\frac{b^6}{c^6} - \&c.\right\}.$$

If we introduce zonal harmonics as coefficients, this becomes

$$\omega = 2\pi\left\{\frac{1}{2}\frac{b^2}{c^2}P_1 - \frac{1.3}{2.4}\frac{b^4}{c^4}P_3 + \frac{1.3.5}{2.4.6}\frac{b^6}{c^6}P_5 - \&c.\right\}.$$

This is now the potential at any point in space where $b < c$.

If there be a second circular circuit of radius β on the same axis, we may suppose it replaced by a magnetic shell bounded by the current and lying on the sphere, with centre at the centre of the first current, the radius of the sphere being c.

This shell may be considered to consist of two layers of

matter of equal and opposite densities, μ and $-\mu$, at distances c and $c+dc$ from the centre. The potential on the second layer is

$$\iint \mu\omega dS,$$

where the integration is taken over the shell. The potential on the second layer is

$$-\iint \mu\left(\omega + \frac{d\omega}{dc}dc\right)dS,$$

the sum being

$$-\iint \mu\frac{d\omega}{dc}dc\,dS\,;$$

but since the strength $=1$, $\mu dc=1$, and we have the mutual potential

$$M=-\iint\frac{d\omega}{dc}dS.$$

Replacing the element dS by $c^2d\mu\,d\phi$, the limits will be for ϕ from 0 to 2π, and for μ from 1 to μ.

Integrating with respect to ϕ, and remembering that c is constant in integrating for μ, we have

$$\begin{aligned}M&=2\pi c^2\int_\mu^1\frac{d\omega}{dc}d\mu\\ &=-4\pi^2c^2\left\{\frac{b^2}{c^3}\int_\mu^1 P_1d\mu-\frac{1.3}{2.4}\cdot 4\frac{b^4}{c^5}\int_\mu^1 P_3d\mu+\&c.\right\}.\end{aligned}$$

But we have the relation for zonal harmonics,

$$\int_\mu^1 P_nd\mu=\frac{1-\mu^2}{n(n+1)}\frac{dP_n}{d\mu}.$$

Substituting, we obtain

$$\begin{aligned}M=-4\pi^2(1-\mu^2)\left\{\frac{b^2}{2c}\frac{dP_1}{d\mu}-\frac{1.3}{2.4}\frac{b^4}{3c^3}\frac{dP_3}{d\mu}\right.\\ \left.+\frac{1.3.5}{2.4.6}\frac{b^6}{5c^5}\frac{dP_5}{d\mu}-\&c.\right\}.\end{aligned}$$

The following are the values for the coefficients (Ferrer's 'Spherical Harmonics,' p. 23), both in terms of μ and when

we substitute $\mu^2=1-\frac{\beta^2}{c^2}$:—

$$\frac{dP_1}{d\mu}=1,$$

$$\frac{dP_3}{d\mu}=\frac{3}{2}(5\mu^2-1)=\frac{3}{2}\left(4-5\frac{\beta^2}{c^2}\right),$$

$$\frac{dP_5}{d\mu}=\frac{15}{8}(21\mu^4-14\mu^2+1)=\frac{15}{8}\left(8-28\frac{\beta^2}{c^2}+21\frac{b^4}{c^4}\right),$$

$$\frac{dP_7}{d\mu}=\frac{3003\mu^6-3465\mu^4+945\mu^2-35}{16}=\frac{448-3024\frac{\beta^2}{c^2}+\&c.}{16},$$

$$\frac{dP_9}{d\mu}=\frac{109395\mu^8-180180\mu^6+90090\mu^4-13860\mu^3+315}{128}$$

$$=\frac{5760+\&c.}{128}.$$

Substituting these values and putting $c^2=a^2+\beta^2$,

$$\therefore\ M=-4\pi\frac{\beta^2b^2}{a^3}\left\{\frac{1}{2}-\frac{3}{4}\frac{b^2+\beta^2}{a^2}+\frac{15}{16}\frac{b^4+3b^2\beta^2+\beta^4}{a^4}\right.$$
$$\left.-\frac{35}{32}\frac{b^6+6b^4\beta^2+6b^2\beta^4+\beta^2}{a^6}+\&c.\right\},$$

The more useful form is obtained by retaining c. If we take the two circles of equal radius (i. e. $b=\beta$), we obtain

$$M=-4\pi^2\frac{b^4}{c^3}\left\{\frac{1}{2}-\frac{3}{4}\frac{b^2}{c^2}+\frac{15}{8}\frac{b^4}{c^4}-\frac{35}{8}\frac{b^6}{c^6}+\frac{2835}{256}\frac{b^8}{c^8}-\&c.\right\}.$$

XXVI. *On a new Form of Resistance-Balance adapted for comparing Standard Coils. By* J. A. Fleming, *D.Sc. (Univ. Lond.), Scholar of St. John's College, Cambridge**.

[Plate XIX.]

1. The British Association Committee on electrical standards concluded their valuable labours on the unit of resistance by constructing copies of the selected standard. Certain of these coils, some fourteen in number, are at present preserved in the Cavendish Laboratory, Cambridge. It is

* Read December 13th, 1879.

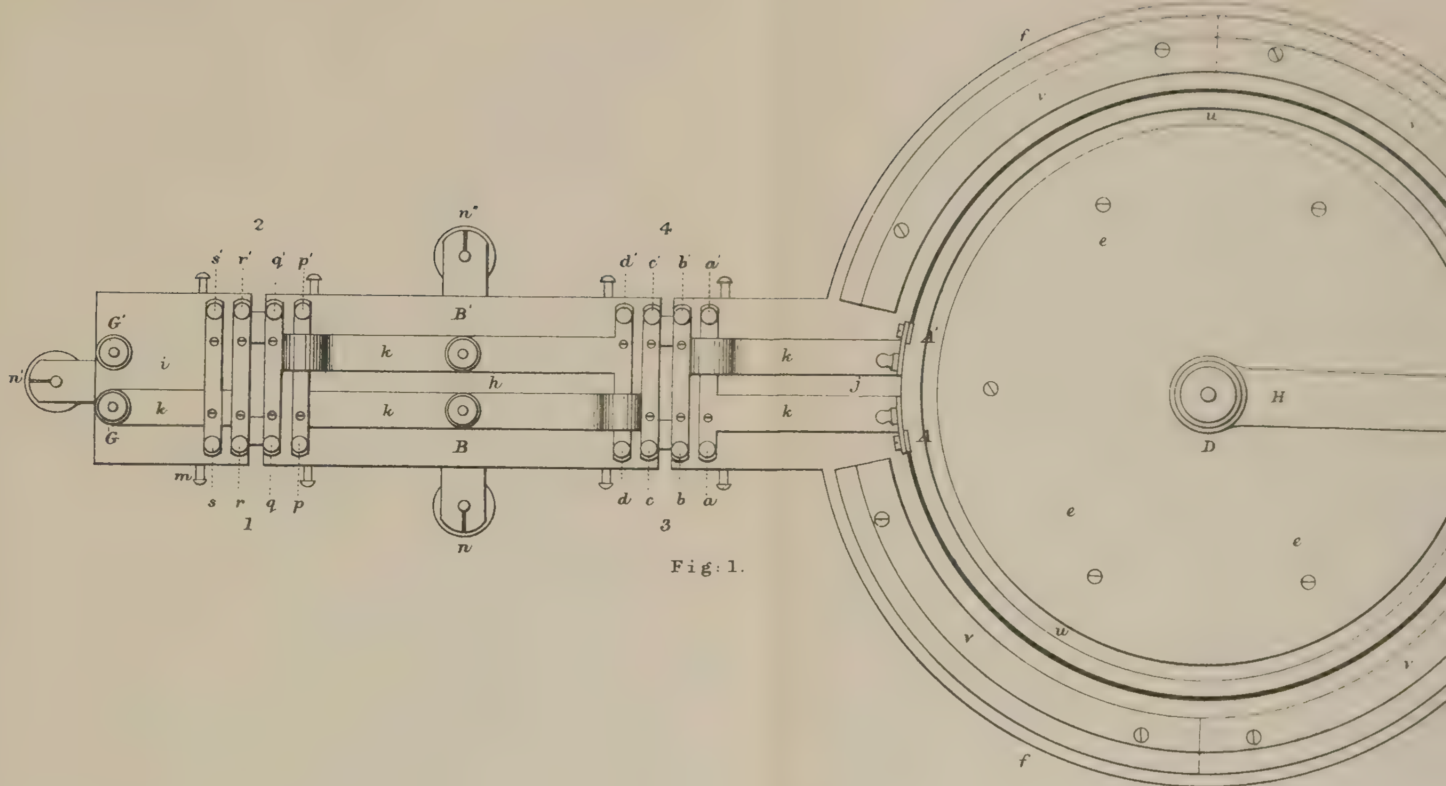


Fig: 1.

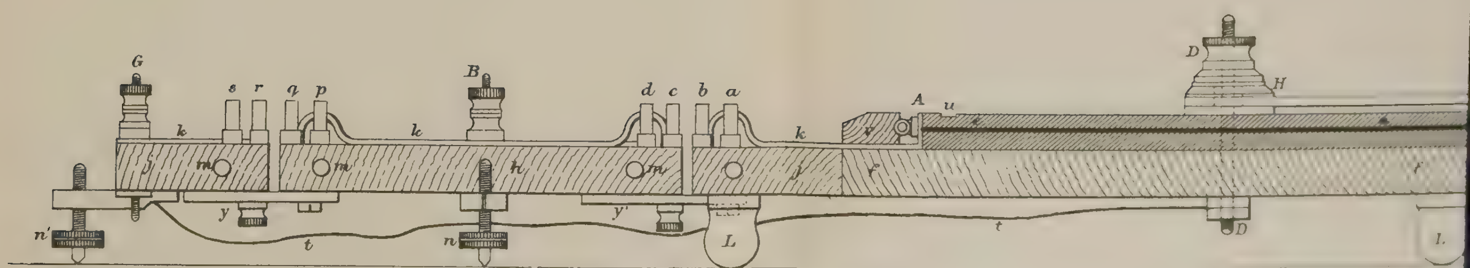
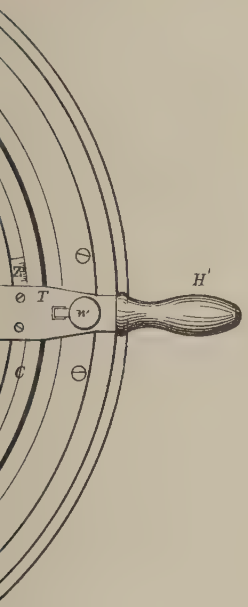


Fig: 2.

Scale for Figs 1, 2 & 3. ¼ Fu



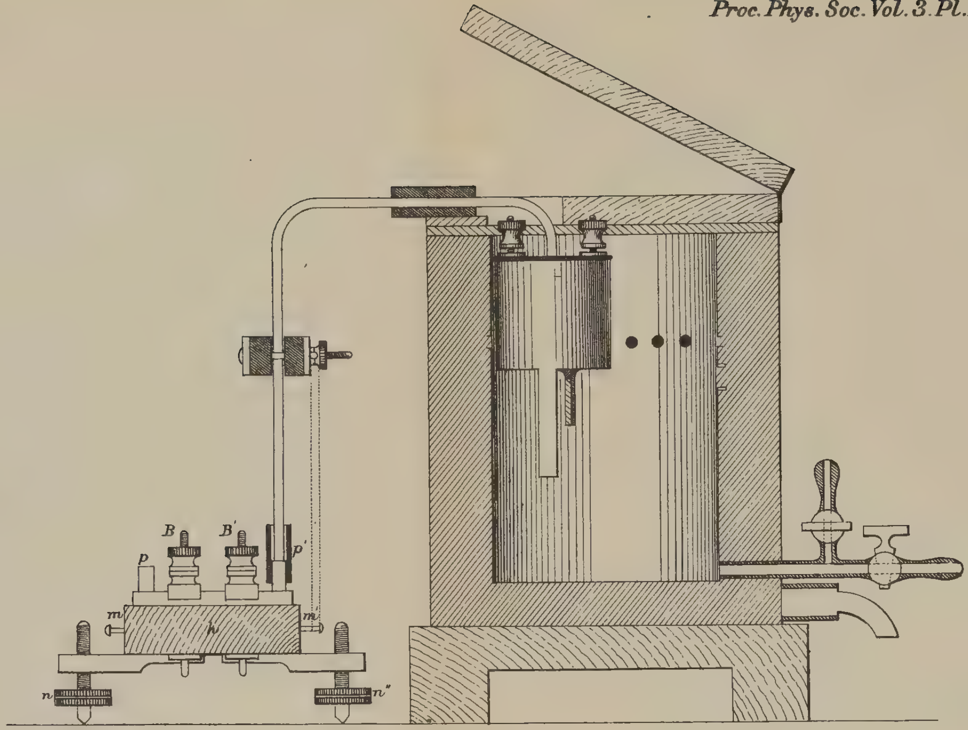
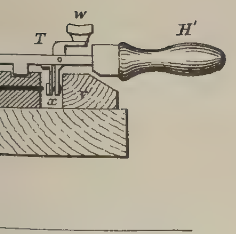


Fig. 3.



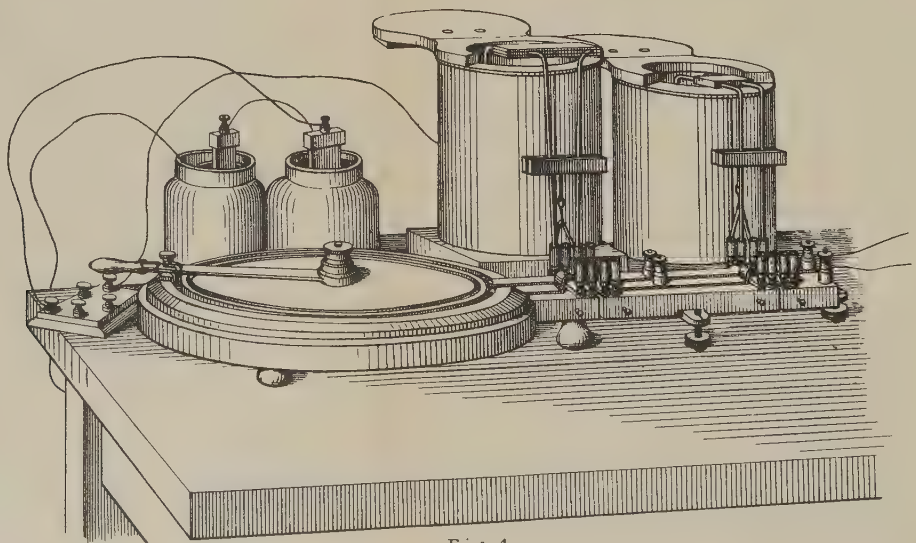


Fig: 4.

Size

important that these coils, which consist of wires of various alloys, should be from time to time carefully compared together in order to determine whether the ratio of their resistances at definite temperatures remains the same*. Observations ought also at the same time to be made of the temperatures at which they agree, and also of their coefficients of variation of resistance with the temperature. In using for the purpose the ordinary form of divided-metre bridge, several objections present themselves which render it a tedious process to determine accurately the difference in the resistance of two coils at different temperatures, and hence to deduce their variation-coefficients. It seemed, on consideration, that a modification of the usual form of Wheatstone's bridge would render these processes more expeditious and at the same time more accurate. It is the object of the present paper to describe a form of resistance-balance which has been recently constructed for the Cavendish Laboratory, and which experience shows to have several decided advantages over the old form.

2. *Description of the Resistance-balance.*—A circular disk of mahogany 18 inches in diameter and about 1 inch thick (f) (Pl. XIX. figs. 1 and 2) stands upon three short feet L. Upon this, and concentric with it, is screwed down a disk of ebonite 14 inches in diameter and $\frac{3}{4}$ of an inch thick (e). This ebonite disk has a semicircular groove turned in its circumference. The circular wooden base extends on one side into a narrow rectangle (j) 4 inches wide and of the same thickness as the disk. To this are connected two other rectangular pieces (h, i,) which are joined together by slotted brass bars (y, see fig. 2) underneath, in such a manner as to permit the two intervals to be made wider or narrower at pleasure. This promontory is of wood, of the same material and thickness as the disk f, and is supported and levelled by three levelling-screws n, n', n''. Through the centre of the ebonite disk passes a brass centre-pin D D′ (fig. 2), on which is centred a brass arm, H H′, capable of

* A detailed and most careful comparison of these coils was made by Prof. G. Chrystal and Mr. S. A. Saunder in 1875; and their Report is printed *in extenso* in the Report of the British Association at Glasgow in 1876. This is the most recent occasion on which these coils have been examined.

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revolving round just clear of the disk. Beneath the arm, and soldered to it, is a short brass spring x, which depends vertically downwards. This spring carries at its extremity a small prism of platinum-iridium with one edge vertical and turned inwards. In the groove turned in the disk e is stretched a platinum-iridium wire about $\frac{3}{32}$ of an inch in diameter. The wire extends round about $\frac{25}{27}$ of the circumference, and is about 39 inches long; and the groove is of such a size that the wire lies with exactly half its thickness imbedded in it. This wire is represented by the thick black line A C A′ in fig. 1. The ends of this wire are soldered to copper strips k, k. On the wood rectangles j, h, i is fastened an arrangement of longitudinal copper strips, k, k, which connect together eight transverse square copper bars in the manner shown in fig. 1. On the ends of these transverse bars are fixed vertical copper pins $\frac{5}{16}$ of an inch in diameter and $\frac{3}{4}$ of an inch high. On these pins are slipped short lengths of india-rubber tube which extend beyond the pins, so that they form small cups about 1 inch deep, p' (see fig. 3). The top of the copper pin is well amalgamated with mercury, and forms the bottom of the cup. These cups are filled about a quarter full of mercury. On the longitudinal strips of copper are fixed three binding-screws B, B′, G; and a fourth (G′) simply goes through the wood, and is connected by a wire t underneath the baseboard with the centre-pin D, and is therefore in metallic connexion with the spring x. The battery is connected with the terminals B, B′, and the galvanometer with the terminals G, G′. To the arm H H′ is adapted a trigger, T, of such shape that when the button w, which is of ebonite, is pressed down, the spring x, carrying the platinum-iridium knife-edge, is bent inwards until it touches the wire strained round the circumference of e. The arm carries a vernier N, which travels round sunk in a shallow groove in the face of the ebonite disk; and the ebonite is graduated on the face on the margin of the groove. The graduations are cut into the ebonite, and then rubbed over with powdered chalk mixed with gum and water. This gives a graduation very legible and pleasant to look at. The length of the wire is just one thousand divisions; and the vernier enables these to be divided into tenths. The zero of graduation is so placed that, when the pointer of the vernier

reads zero, the knife-edge on the spring x is exactly opposite the extremity of the platinum-iridium wire.

It is thus clear that the revolving arm carrying its knife-edge can be moved round so that, on pressing the trigger-button w, the knife-edge makes contact at any point of this wire, and thus connects this point with the terminal G′.

This part of the arrangement answers to the sliding block and piston-contact piece of the ordinary divided-metre bridge.

3. *Method of using the Balance.*—Let now two resistance-coils of about equal resistance be provided, and let the coil-terminals of one coil be placed in the mercury-cups p and r, and those of the other be placed in q' and s'. And let two more coils be taken of not very unequal resistance which it is desired to compare with each other, let the terminals of one be placed in the mercury-cups a and c, and those of the other in b' and d'. It will then be seen that if a battery be connected with B B′, and a galvanometer with G G′, that we have the usual Wheatstone's bridge arrangements (see fig. 5 on page 179, which gives a diagram of the connexions). Two quart Leclanché cells are best suited for ordinary use. If a more powerful battery is used, there is danger of heating the platinum-iridium wire, and so expanding it that it may slip down out of its groove.

The coils in the intervals between the cups p and r and q' and s' form two branches; and the coil in the interval between a and c, together with the resistance of the platinum-iridium wire round to the place where the spring x touches it, forms the third branch, whilst the coil in the interval b' d', together with the remainder of the wire, forms the fourth. The "bridge"-wire consists of the arm H H′ and the wire under the base-board together with the galvanometer inserted between G and G′. By moving round the arm H H′ and pressing the button w, we can find a position where there is no current through the galvanometer. The copper strips k k are made of copper so thick that their resistance is practically nothing. Having established a balance between the conductors and read the vernier, the next operation is to lift up the legs of the coil which were inserted in the cups a and c and drop them into the cups b and d. Likewise a similar change is effected on the other side; the terminals of the coil inserted

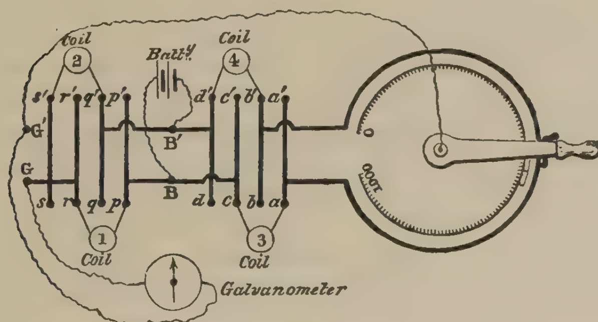
in b' and d' are changed to a' and c'. An examination of the connexions as shown in fig. 1 will show that the result of the operation is as if the coils had *changed places* whilst preserving their former connexion. Now let the arm be moved round and a fresh position of equilibrium found by pressing the trigger and reading the vernier. A little consideration will show that the difference of these readings gives the difference between the resistances of the coils in terms of a length of the bridge-wire; for the amount by which one coil exceeds the other in resistance is equal to the resistance of that part of the bridge-wire included between the two readings*. In order to render this method of determining the difference of the two coils practicable, the platinum-iridium wire must be exceedingly uniform in resistance, or else a Table of calibration will have to be made. Great pains were taken to procure a length of wire as uniform in size and resistance as possible; and considerable care was taken, in laying the wire in its groove, not to strain it in any way. It lies evenly in its groove, just sufficient tension being put upon it to keep it in its place. The whole resistance of the wire from end to end is not far from $\frac{1}{20}$ of an ohm at about 15° C.

The wire was carefully calibrated by measuring the difference in the resistance of two pieces of thick brass wire of such lengths that the difference of their resistances was about equal to that of thirty divisions of the bridge-wire; and this difference was measured at about a hundred different equidistant positions all along the bridge-wire, and found to be so nearly the same that no Table of calibration was deemed requisite. To protect the bridge-wire from injuries, as well as to preserve it from being heated by radiation from surrounding bodies, a wooden ring, $v\,v$, is fastened down on the base-board. The ring is $1\frac{1}{4}$ inch wide and $\frac{3}{4}$ inch deep; and its internal

* This method of obtaining the difference of two resistances in terms of a length of the calibrated bridge wire was suggested by Prof. G. C. Foster, F.R.S., in a paper read before the Society of Telegraphic Engineers, May 8, 1872. In this paper is given an account of the method of calibrating a wire. It is obvious, without any further proof, that if the coil placed in a and c exceeds in resistance that placed in b and d, then on exchanging them, since the united resistance of coils and bridge-wire remains the same, that the contact knife-edge must be moved back along the bridge-wire by a length exactly equal in resistance to the excess of one coil over the other.

diameter is 1 inch greater than that of the ebonite disk. The wire, therefore, lies hidden away on the side of a square-sectioned circular tube; and, furthermore, a shield of cardboard faced with tinfoil lies upon the face of the disk e, extending just beyond the ring. An aperture is cut in this shield to permit the passage of the trigger, as well as to allow the vernier to be read. By this means the wire is not only out of sight, but out of reach of all radiation as well as mechanical injury.

Fig. 5.



4. *Method of determining the Variation-coefficients of Coils.*—To determine the variation-coefficient of any given coil we proceed as follows:—Three other coils are provided, two of them nearly equal in resistance, which we will call 1 and 2. A third coil, 3, must be taken whose resistance is nearly equal to that of 4, the coil whose variation-coefficient is desired (see fig. 5). The terminals of 3 are inserted in the mercury-cups a and c, those of 4 in b' and d', those of 1 in p and r, and those of 2 in q' and s'. Now the operation to be conducted is to keep the coils 1, 2, and 3 at a fixed temperature, and to keep 4 successively at two known temperatures, differing by about 15° Cent., and to obtain the difference of the resistances of 3 and 4 at these two temperatures. The difference of these differences, divided by the difference of the temperatures, is the mean coefficient of variation of resistance between these temperatures. The chief difficulty to be contended with is that of keeping the temperature of the coils constant during the operation, and of ascertaining what that temperature is; for, as Prof. Chrystal has remarked in his report (Brit. Assoc.

Report, 1876), it is not easy to tell whether the temperature of the water in which the coil rests is identically the same as that of the wire, since the latter is imbedded in a mass of slowly conducting paraffin. To reduce as far as possible the difficulty of keeping the coils at a constant temperature, they are placed in water-vessels made of zinc (see fig. 3, Pl. XIX.). These water-boxes are composed of two cylindrical vessels—an outer case 9 inches high and 8 inches in diameter, and an inner one of lesser size; the two are connected at the top, so that they form a sort of jar with hollow sides and double bottom. This interspace forms an air-jacket. Around the inside vessel near the top is a row of small holes; and two tubes communicate at the bottom—one with the inner vessel, and the other with the annular interspace. The top is closed by a wooden lid with apertures for thermometer and stirrer. Water can be made to flow from the supply-pipes into the inner vessel; it rises up and overflows through the holes, and drains away down the interspace and out by the other pipe. The bodies of the four coils are placed in four water-boxes of this description; and water from the town mains being sent in a continuous stream through all four water-boxes, the coils are rapidly brought to and maintained at a known temperature. Any desired temperature can be given to one coil by leading warm water from a cistern into its vessel. The annular air-filled space renders the rate of cooling very slow. Hence the coils, once at the desired temperature, can easily be kept there. Fig. 4, Pl. XIX., gives a sketch of the arrangement, two of the water-boxes being removed to show the connexions.

The advantage of the somewhat complicated arrangement of copper bars will now be seen. We can, without withdrawing the coils 3 and 4 from their water-boxes, and without in any way disturbing the other arrangements, *reverse* the position of the coils 3 and 4 on the bridge, by simply lifting up the legs half an inch and changing the mercury-cups into which they dip. Thus the legs of coil 3 are changed from cups a and c to b and d, and those of coil 4 from b' and d' to a' and c'. This exchange does not occupy more than a few seconds; and hence we can obtain the two readings necessary to give the difference of the resistance of the coils 3 and 4 when they are at different temperatures in a very short time. During this

short time the temperatures of the two coils will not change perceptibly, protected as they are by an air-jacket.

In the ordinary form of straight bridge there is considerable trouble in exchanging the coils, because the water-vessels have to be moved and the mercury-cups readjusted; and all this time the coils are cooling; so that the two readings are never made under the same circumstances as regards temperature. Beginning, then, with all four coils at the same temperature, we take the difference between 3 and 4. To get them all at the same temperature, water from the town mains is allowed to circulate through the system for half an hour. At the end of this time the difference of 3 and 4 is taken; and several readings are taken at small intervals of time to see if the temperatures are constant. This being done, the temperature of coil 4 is raised by the introduction of warm water until it is about 15° above that of coil 3. It is best to raise the temperature about 20° above the other at first, and keep it there for 20 minutes, and then let it fall very slowly. In this way coil and water cool together, and an equilibrium of temperature is established between them. The difference between 3 and 4 is again taken; and from these two readings we have, as seen above, the mean variation-coefficient between the two temperatures. Another method, which would probably be a more accurate one, for obtaining the mean coefficient of variation between 0° C. and 15° C. would be to wait until the temperature of the water in the town mains was about 15° C., and then to keep three of the coils at that temperature, and to cool the fourth by means of ice to zero. If then all four were kept at 15° and the observations repeated, we should have the means of finding the variation-coefficient of the fourth coil between 0° and 15°. Prof. Chrystal, in his report, threw out the suggestion that resistance-coils should have a thermoelectric couple attached to them, one junction being buried in the heart of the paraffin surrounding the wire, and the other outside. This has been tried in some coils recently made, and proves a satisfactory method of ascertaining the equilibrium of temperature between the wire and the water.

Another source of error in the ordinary methods arises from uncertain or variable resistances at the mercury-cups. It is important that the copper legs of the coil-terminals should

press very firmly against the tops of the copper pins on which the india-rubber-tube cups are fixed. To ensure this, the plan adopted is to fasten on the coil-legs an ebonite clamp. Along the edge of the wooden promontory, $j\,h\,i$ (fig. 1), are put brass pins m; and by means of steel spiral springs fixed to these and attached to the clamps the coil-legs are pressed down very firmly (see fig. 3). The ends of the pins which carry the india-rubber cups and the ends of the coil-legs being well amalgamated, we get, when they are thus firmly pressed in contact, a very good joint, and one whose resistance is small and constant. If the clamps are not used, then one leg may get lifted up a little, and thus a short length of mercury interposed, which leads to an error in a reading.

5. *Example of a determination of the Variation-coefficient of a Coil.*—The whole resistance of the platinum-iridium wire is very nearly 0·0512 of an ohm, or not far from $\frac{1}{20}$ of an ohm, at about 15° Cent. As the whole length can be divided by the vernier into 10,000 parts, this gives as the value of $\frac{1}{10}$ of a division $\frac{1}{2000000}$ of an ohm.

The unit in the following example is $\frac{1}{10}$ of a division. To secure the greatest accuracy of measurements a low-resistance galvanometer must be used. I am in the habit of using one having a resistance of about half an ohm. The image of a wire strained across a slit is reflected on a scale in the usual way, and read at a distance by means of a telescope. This galvanometer will give an indication, when used with precautions, due to a difference of one tenth of a division when comparing two ohm coils. But as the temperature can hardly be measured with certainty to within less than $\frac{1}{20}$ of a degree, this alone renders such refinement of reading nugatory, in the absence of better methods of ascertaining with certainty the real temperature of the wire.

Two coils were compared. Call them F and K. K is the coil whose variation-coefficient is required.

I. *Difference of Resistance of Coils* F *and* K *at* 11° *Cent.*

	Bridge-readings.		Difference.
Exp i.	5000	4955	45
Exp. ii.	5000	4954	46
Exp. iii.......	5000	4955	45

The first column gives the number of experiment, the second the reading with the coils F and K in one position on the bridge, the third when F and K are reversed or have exchanged places on the balance; and the fourth gives the difference of their resistances at 11° C. in units of the bridge-wire.

II. *Difference of Resistance of Coils* F *and* K *at* 28°·2 *Cent.*

	Bridge-readings.		Difference.
Exp. i.	5439	4492	947
Exp. ii.	5442	4497	945
Exp. iii.	5440	4490	950

As before, the fourth column gives the difference of F and K at 28°·2 C. Taking the mean difference at 28°·2 C. to be 947 units, and that at 11° C. to be 45 units, we have

$$\frac{947-45}{28{\cdot}2-11}=52{\cdot}4 \text{ units}$$

as the mean variation-coefficients between 11° C. and 28° C. in units of bridge-wire. Since the coils F and K are approximately ohm coils, this gives as the variation-coefficient of the coil K ·0262 per cent. This coil is of platinum-silver wire. These three determinations occupied about an hour and a half, during which time many more readings were taken, all closely agreeing with the above. The actual measurement of the differences requires but a few moments to effect, the principal expenditure of time being that required to bring the coils to the same temperature as the water.

In conclusion, I may state that this resistance-balance has been constructed in the workshops of the School of Mechanical Engineering at Cambridge, under the direction of Prof. Stuart. Great care was taken in laying on the wire so as to avoid straining it in any way; and the performance of the instrument is consequently very satisfactory. I should also express the fact that this excellent performance is due to the supervising care of Prof. Stuart, who not only supplied several of the details of the construction, but aided, by his valuable suggestions generally, during the process of carrying out my rough designs into a practical form.

Cavendish Laboratory, Cambridge,
December 1879.

XXVII. *A Dispersion-Photometer.*
By John Perry *and* W. E. Ayrton *.

In measuring what is usually termed the power of a light, it is common to have a screen placed at such a distance from the light that its illumination is equal to that which it or another screen receives from a standard candle. Now, if a standard candle is, say, one foot from a screen, an electric light of, say, 6400 candle-power must be placed at the distance of 80 feet from a screen to give the same illumination. That a great distance like this should be necessary, and in a chamber whose walls are supposed to be blackened, in the laboratories of works where electric lights are usually examined, has placed great difficulties in the way of exact determinations of the powers of lights. Our experience in experimenting with electric lights leads us to believe that, but for this difficulty, we should have a vast body of information on the subject of electric lighting, instead of the vague and conflicting statements which fill the scientific journals. Perhaps only those who have made experiments will appreciate fully the great advantages of our having some compact form of photometer. The members of the Society will no doubt see many applications of a compact photometer, such as measuring the light from various parts of the sky for instance, in which ordinary photometric methods are unavailable. Now our instrument will resemble a camera; and it may be turned in all directions.

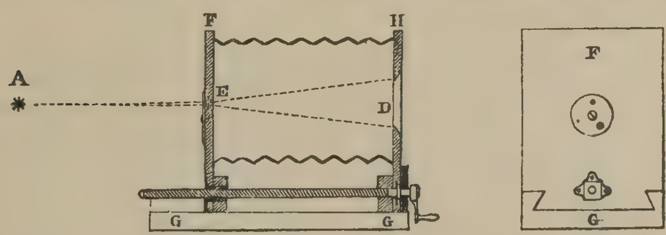
We were delighted to hear from Dr. Guthrie that his principle of a "retention-image photometer," recently communicated to the Society, has proved to be correct quantitatively. The success of his test must be as interesting to physiologists as to physicists. Besides testing his principle by using powerful lights, he will have experiments to make concerning the loss of light in reflection, as we shall have concerning loss in refraction. Probably, however, he does not intend to apply his instrument to powerful lights, on account of the very great difficulty he would meet with in measuring the breadth of the fine slot which would be needed.

In ordinary photometric methods, the rays of light illumi-

* Read December 13th, 1879.

nating unit area of screen, if coming from a powerful source, are contained in a very small solid angle. We use the very simple expedient of causing these rays to fill a much greater solid angle by passing them through a thin concave lens, and in this way obtain the same amount of illumination as before, but on a screen placed at a short distance behind the lens. Thus we not only save space, but prevent a great deal of the absorption which occurs when light passes through air. This absorption is sometimes very great in London. We have, however, absorption in the lens as a disadvantage; but with a very thin lens this may probably be reduced to an almost inappreciable amount.

A is the light to be measured; D is a paper screen illuminated by light passing through the concave lens E. The frame F, which carries the lens, slides on a stand G, which has marked divisions, so that a pointer tells the distance from the focus of the lens to the screen D. The sides of the space F D are of black cloth, like the sides of a folding camera; and the inside is all blackened except the screen D. A circular plate with three round holes of different sizes is in front of the lens. A diaphragm of this kind is not necessary, if we can be assured of there being no reflection from the inside of the box. Beside the instrument, and containing the standard



candle, is a box blackened inside, in one end of which is a screen similar to D. If D is the distance from the light to the lens and d is the distance from the principal focus of the lens to the screen; if δ is the focal length of the lens, then, roughly, a bundle of rays of unit solid angle gets to have, after refraction, an angle

$$\frac{D^2}{\delta^2},$$

as D is always great in comparison with δ. If L is the total light, then the unit angle of incident rays and the unit angle of refracted rays have amounts of light

$$\frac{L}{4\pi} \text{ and } \frac{L}{4\pi} \cdot \frac{\delta^2}{D^2};$$

so that a screen at the distance d from the focus of the lens has illumination of the intensity

$$\frac{L}{4\pi} \cdot \frac{\delta^2}{D^2} \cdot \frac{1}{d^2}.$$

If, now, another screen has the same illumination from a candle whose total light is unity, at the distance D_1 this illumination is

$$\frac{1}{4\pi D_1^2} = \frac{L}{4\pi} \cdot \frac{\delta^2}{D^2} \cdot \frac{1}{d^2};$$

and hence

$$L = \frac{D^2}{D_1^2} \cdot \frac{d^2}{\delta^2}.$$

A double-concave lens, of focal length 1 centimetre, being employed, and d being capable of variation from 40 to 10 centimetres, and D being as much as five times D_1, we can measure in a very small space a light which is from 40,000 to 100 times the standard candle.

We propose to make careful experiments on the absorption in passing through the lens; and by interposing thin plates of glass between the standard candle and its screen to produce there a similar absorption, we have no doubt that measurements may be made with this instrument with much greater accuracy than is possible by the ordinary method.

The power of the light is practically, therefore, proportional to the square of the product of the measurable distances. It will be, of course, unnecessary to measure the variations both in D and in d, as the scale can be so graduated as to give both distances at one reading.

XXVIII. *On Intermittent Currents and the Theory of the Induction-balance.* By Oliver J. Lodge, *D.Sc.**

1. The telephone, considered as a scientific instrument, seems destined to play an important part as a detector of minute currents of rapidly changing intensity; and the general theory of intermittent currents is being brought into prominence by its use.

The equations to which most attention has been hitherto directed have been those relating to the steady flow of a current after the initial inductive or inertia-like effects have subsided; and in arrangements such as the Wheatstone bridge, a double key is commonly used, in order to allow the introductory stage to subside before any observation is taken. The galvanometer is essentially an instrument for measuring steady currents, or for giving the algebraically integrated expression for the total quantity of electricity which has passed in the case of transient currents; that is, $\int_0^\tau i\,dt$, τ being small compared with the period of swing of the galvanometer-needle.

Again, the electrodynamometer has an important use as an integrator of the current without paying attention to sign; that is, its indications give the value $\frac{1}{\tau}\int_0^\tau i^2\,dt$, where again τ, the total duration of the current, is small.

But the telephone-plate has such a very small period of swing that the same τ, which is vanishing compared with the time of oscillation of a needle, may be many times greater than that of the telephone-plate. Moreover the plate is not limited to one mode of vibration, but can have minor vibrations superposed on the fundamental; so that it can enter into the changes going on, and render minute fluctuations audibly apparent which would in more slowly moving detectors be simply merged in the total effect.

Thus a rapidly alternating current (such as the telephone itself produces), which is totally unfelt by a galvanometer, is appreciated by a telephone-plate in its variability—the pitch of the note indicating the number of vibrations, even if they

* Read on the 24th of January, 1880.

are so rapid as to produce a shrill whistle. The telephone, in fact, does not integrate the current, but gives all its fluctuations within certain limits.

The complete theory of the telephone, setting forth precisely on what the loudness of its indications depends, would be a most interesting and important investigation; but if it has been attacked, I am ignorant of it. It seems probable that the loudness of the sounds will be found to depend upon the amplitude of the vibrations of the plate and upon their velocity conjointly—in other words, both on the total change of the current and on the rate at which the change takes place; *i. e.*, in the symbols hereafter to be used, that the loudness is a direct function of $j\,\frac{dj}{dt}$. I shall not, however, assume anything of this sort, but shall content myself with simply finding the value of the current j as a function of the time, leaving the rest to be done subsequently.

It is quite true that the telephone is only an indicator and not a measuring-instrument; but so many null methods can be devised which permit measurements to be made with a simple detector, that it is probable that it will have important applications in this capacity also. And hence I think the general theory of intermittent currents, or of currents in the variable stage, will come into more prominence than hitherto.

The induction-balance furnishes an illustration. Dove made experiments with it; and Felici established the laws of current induction by its aid: but its power as an instrument of research was never appreciated till Prof. Hughes applied to it an intermittent current and a telephone.

Faraday interposed blocks of copper between a primary coil and a secondary connected with a galvanometer, and was surprised to find that the effect at make and break was precisely the same with the copper as without it*. It was, however, afterwards found that the physiological effects were very different, being much less when the copper was present—thus proving that though the copper did not affect the integral flow of electricity, yet it greatly affected the time during which that flow took place.

* Exp. Res. arts. 1721, 1735.

Dove repeated Faraday's experiment more perfectly by means of an induction-balance, and showed that no non-magnetic media produced any effect appreciable by a galvanometer.

Thus may numerous phenomena be overlooked with a galvanometer which with a telephone become obtrusively evident.

2. My attention was more particularly directed to the subject by an observation which was made by Mr. W. Grant, assistant in the Physical Laboratory, University College, and which I have his permission to describe.

The intermittent current from a mouth-telephone, or the induced current from a clock-ticking microphone and coil, is sent through a long helix of wire wound upon a bobbin, with another similar but quite disconnected wire wound alongside it. A telephone and ear are also arranged in the circuit of the first wire, and the loudness of the sound observed. The disconnected wire, which is wound on the same reel as the first, now has its two ends joined up so that it itself forms a closed circuit: the loudness of the sound is thereby considerably increased. The secondary circuit is broken again, and the sound again becomes faint. The strengthening on closing the secondary circuit is so great, indeed, that short-circuiting the primary so as to shunt out all its resistance, scarcely produces any additional strengthening effect. In other words, a moderate resistance (several ohms) may be thrown into a telephonic circuit without the slightest appreciable weakening of the effect, provided a second wire coiled alongside the first be arranged so as to form a closed circuit. If a second telephone be put in circuit with this second wire, it will give about the same indications as the other; and this may sometimes be a good way of working two telephones.

The rough general explanation is, of course, not far to seek. The extra currents, which at first oppose the primary, are able to form in the secondary when its circuit is closed; and hence the changes in strength of the primary are more rapid, and therefore more complete.

An obvious modification is, to arrange both speaking and hearing telephone in circuit with both the parallel coiled wires in series, first, so that the currents flow the same way in both wires, and then with one wire reversed; so that in

the first case the self-induction is a maximum, and in the other almost zero.

Any one who tries this experiment will at once appreciate the very decided difference in the telephonic indications. In fact, resistance is of much less consequence in a telephone-circuit than is self-induction. This fact obviously explains the discrepancy which has been observed between experiments on resistance-bobbins thrown into a telephonic circuit in a laboratory, and the same resistance when forming an actual line. The resistance-coils are wound so as to have no self-induction; and they accordingly weaken the effect very little. It is surprising how many thousand ohms can be thrown into a talking circuit without stopping a conversation. But if the bobbins were wound all in one direction, they would be still worse than a line-resistance.

3. It is plain from this that the only right method of arranging a telephone-line is to have two wires running close together—one for the direct, the other for the return current. This arrangement has, I believe, been used in order to avoid induction-effects from battery-currents in neighbouring wires; but it has also the great advantage of diminishing the self-induction of the wire itself. It is, however, a question how far the increase in electrostatic capacity would militate against this advantage in a very long land-line. It is certain to be injurious as far as it goes; but if the wires are very thin, their electrostatic capacity will be but small.

In the case of a cable the double wire would have decidedly less capacity than a single wire; and the capacity diminishes rapidly with the size of the wire. Hence it appears that a telephone-cable should have a pair of exceedingly fine insulated wires of the very highest conductivity pretty close together in a large insulating sheath.

I need hardly say, perhaps, that no advantage will be gained by attempting to do away with the self-induction of the telephone-bobbin itself by means of a second wire wound alongside the first and forming a closed circuit—because the intermittent currents induced in this would be nearly as strong as those in the direct circuit, and, being always in an opposite direction, will nearly neutralize their intermittent effect on external coils or on the telephone-plate.

I propose now to work out the theory of the induction-balance, and of one or two other cases of intermittent currents, as completely as possible without taking into account the electrostatic capacity of the wires. It would be very interesting to obtain the perfectly general equations; but they are very complex, and appear to be beyond me at present.

General Theory of Current-Induction.

4. The general theory of the establishment of a current in a number of circuits of known resistance, in one or all of which a given electromotive force begins to act, was begun by Helmholtz and Thomson, who independently showed that the fact of the existence of induced currents, discovered experimentally by Faraday, as well as the laws of the same, could be deduced mathematically from the electromagnetic attractions of Oersted and Ampère by means of the principle of the conservation of energy. The complete theory, it is needless to say, is to be found in Maxwell's 'Electricity,' and also in that admirable digest of the whole subject, Professor Chrystal's article in the *Encyclopædia Britannica.*

Suppose there are n circuits of resistances $r_1, r_2, \ldots$, in each of which "impressed" electromotive forces $E_1, E_2 \ldots$ begin to act at the same instant; then the strengths of the currents $i_1, i_2 \ldots$ in the several circuits at any time t afterwards are to be found by the following set of n equations—

$$\left.\begin{aligned} E_1 - r_1 i_1 &= \frac{d}{dt}(M_{11}i_1 + M_{12}i_2 + \ldots + M_{1n}i_n), \\ E_2 - r_2 i_2 &= \frac{d}{dt}(M_{21}i_1 + M_{22}i_2 + \ldots + M_{2n}i_n), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ E_n - r_n i_n &= \frac{d}{dt}(M_{n1}i_1 + M_{n2}i_2 + \ldots + M_{nn}i_n), \end{aligned}\right\} \quad . \; . \; (1)$$

where M_{lk} is a double integral, called the coefficient of mutual induction between the circuit l and the circuit k.

Induction-balance.

5. Now in the induction-balance as ordinarily arranged there are three such circuits, with the coefficients M all constant, and two of them without any electromotive force. The first, or battery-circuit, consists of a battery and two equal

coils, and it also contains some kind of intermittent break or microphone; the second is the telephone-circuit, and consists of two equal coils and a telephone; and the third is complete within the coin or other piece of conducting matter in the neighbourhood of one pair of the coils of the other circuits. The general value of the telephonic current at any instant after a make has occurred in the primary would therefore be obtained by the solution of a linear differential equation of the third order with constant coefficients.

But the problem may be simplified by observing that the "coin" is not introduced until a perfect "balance" has been obtained between the first and second circuits—that is, until they are conjugate to one another so that $M_{12}=0$. Moreover the coefficients of induction between the two coiled circuits and the coin are ordinarily sufficiently small for their squares to be neglected; in which case, as will be shown more fully later (sections 12 and 13), the above set of equations reduces to the following—

$$M_{11}\frac{di_1}{dt}+\quad 0 \quad+\quad 0 \quad+r_1 i_1=E,$$

$$0 \quad+M_{22}\frac{di_2}{dt}+M_{23}\frac{di_3}{dt}+r_2 i_2=0,$$

$$M_{31}\frac{di_1}{dt}+\quad 0 \quad+M_{33}\frac{di_3}{dt}+r_3 i_3=0,$$

or, as I shall for greater convenience write them,

$$\left.\begin{aligned} L\frac{di}{dt}+Ri&=E,\\ l\frac{dj}{dt}+m\frac{dk}{dt}+rj&=0,\\ \lambda\frac{dk}{dt}+\mu\frac{di}{dt}+\rho k&=0, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

where the three coefficients of self-induction are written L, l, and λ (λ being very small); and where the battery-current is i the telephone-current is j and is small, and the current in the coin is k; m signifies the effect of the coin-current on the telephone-circuit, and μ the effect of the primary circuit on the coin. The effect of the primary on the secondary (M) is, as already explained, supposed to be 0; that is, balance is supposed to have been obtained before the insertion of the coin. ρ, or

the resistance which the current circulating in the coin experiences, is a quantity which is likely to be several times larger than the specific resistance of the material (*i. e.* the resistance of a centimetre cube); but for ordinary coins it is a number of the same order of magnitude. It must be remembered that R and L stand for the resistance and self-induction of the *whole* of the battery-circuit, and, similarly, r and l include the telephone-coil as well as the other two coils of the telephone-circuit.

Now the only difficulty in solving these equations consists in the varying resistance of the battery-circuit, in which a contact-breaker, or clock-ticking microphone, or some other arrangement for producing an intermittent current is inserted. I shall therefore leave the consideration of the induction-balance for the present, and examine the case of a primary circuit in space by itself and having a break and make in some part of it. Then we will consider the case of an intermittent primary in the neighbourhood of a closed secondary; and after this it will be easy to apply our results to the induction-balance.

On the Law of Variation of a Battery-current in a solitary Circuit.

6. Let a battery of constant electromotive force E act in a circuit whose total resistance may be *suddenly* changed from the value R to the value S, and let L be the coefficient of self-induction of this circuit. Then the current i at any time t after the change of resistance has occurred is to be found from the equation

$$\mathrm{L}\frac{di}{dt}+\mathrm{S}i=\mathrm{E}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

with the initial condition $i=\frac{\mathrm{E}}{\mathrm{R}}$ when $t=0$. This gives us in the integral form

$$i=\frac{\mathrm{E}}{\mathrm{S}}\left(1+\frac{\mathrm{S}-\mathrm{R}}{\mathrm{R}}e^{-\frac{\mathrm{S}}{\mathrm{L}}t}\right). \quad . \quad . \quad . \quad . \quad (4)$$

Putting $\mathrm{R}=\infty$, we get the well-known expression for the current at "make,"

$$i=\frac{\mathrm{E}}{\mathrm{S}}(1-e^{-\frac{\mathrm{S}}{\mathrm{L}}t}). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Putting S very large, we get an expression for the current at

partial break, provided there is no extra-current spark at the surface of separation,

$$i=\frac{\mathrm{E}}{\mathrm{R}}e^{-\frac{\mathrm{S}}{\mathrm{L}}t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It will not do to put $\mathrm{S}=\infty$, because it is impossible to make the resistance *suddenly* infinite, *i. e.* to stop the current instantaneously.

7. It is difficult to get a reasonably correct expression for the value of the current at break. We may suppose that, by separation of two portions of the circuit, the resistance is suddenly changed from R to a quantity which would be S if the current instantaneously ceased, but which is much less than S for a very short time, owing to the heat generated by the current itself. The temperature of the spark at any time t after the break is to be found, we may suppose, from such an equation as this,

$$mc\theta=\int_0^t (ri^2-\mathrm{H}\theta)dt,$$

where H is the cooling-constant. The mode in which the resistance of air changes with temperature is unknown; so we may assume any simple and not improbable law, such as that the decrement of resistance is proportional to the increment of temperature, or

$$dr=-k\,d\theta,$$

where k is a constant. From these two equations r has to be obtained as a function of the current i and of the time t; and then its value has to be substituted for S in equation (3). As a first approximation, we may imagine the time too short for cooling, or make $\mathrm{H}=0$. Then we get

$$r=\mathrm{S}e^{-k'\int_0^t i^2dt};$$

an expression which, as a coefficient in equation (3), to me seems quite unmanageable. I will therefore assume for the future that the change in the battery-circuit resistance is always made suddenly from one finite value to another, no spark or current across air-spaces being produced.

8. *Suppose the resistance of a battery-circuit oscillates rapidly between the values* R *and* S, *each change being made suddenly*

and lasting for the short time τ; what is the strength of the current after a few seconds?

This may be assumed to be the sort of thing that happens in a microphone or other arrangement for producing an intermittent current.

The strength of the current before the vibration begins is

$$\dot{i}_0 = \frac{E}{R};$$

at the end of the first period τ the strength is, by equation (4),

$$\dot{i}_1 = \frac{E}{S} + \frac{E(S-R)}{SR} e^{-\frac{S}{L}t};$$

at the end of the second period τ the current is

$$\dot{i}_2 = \frac{E}{R} - \frac{E(S-R)}{SR} (e^{-\frac{R}{L}\tau} - e^{-\frac{R+S}{L}\tau}),$$

and so on—

$$i_4 \text{ being } \frac{E}{R} - \frac{E(S-R)}{SR} (\text{exp.}) \{(R) - (R+S) + (2R+S) - (2R+2S)\} \left(-\frac{\tau}{L}\right)$$

and

$$i_5 \text{ being } \frac{E}{S} + \frac{E(S-R)}{SR} (\text{exp.}) \{(S) - (S+R) + (2S+R) - (2S+2R) + (3S+2R)\} \left(-\frac{\tau}{L}\right)$$

Accordingly, after a very large number of vibrations (which will be accomplished in a few seconds) the current is either

$$\dot{i} = \frac{E}{S} + \frac{E(S-R)}{SR} \left(\frac{1 - e^{-\frac{R}{L}\tau}}{1 - e^{-\frac{S+R}{L}\tau}} \right) e^{-\frac{S}{L}t}, \quad . \quad . \quad (7)$$

or else the same expression with R and S everywhere interchanged, according as the last change of resistance was from R to S or from S to R, t being the time which has elapsed since the last change.

The law of variation of such a current is therefore just the same as for a simple make and break (4); but the part depending on time is multiplied by a constant fraction always less than unity, and which diminishes rapidly as τ (the period of a semi-vibration of the intermittence) diminishes.

If τ be infinitesimal compared with $\frac{L}{R+S}$, the fraction is $\frac{R}{S+R}$, and hence the current is

$$i=\frac{E}{S}\left(1+\frac{S-R}{S+R}e^{-\frac{S}{L}t}\right),$$

or, as of course t (being less than τ) is itself vanishing,

$$i=\frac{2E}{S+R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence the current tends to approach this constant value if the vibrations are too rapid. Such a current as this is of course inappreciable by a telephone ; but the above would be its galvanometric indication.

This must be something like the state of things in a coil with a too rapid break.

Query, whether any thing of the same sort happens when a battery-current is passed through a vacuum-tube, the intermittence being almost too rapid to be heard by the telephone. Drs. De la Rue and Hugo Müller have put a telephone in circuit with a battery and vacuum-tube, and heard only a faint rustling when the stratifications were steady.

Equation to a continuously Intermittent Current.

9. It would be interesting to obtain an expression for the current when the resistance of the circuit varies *continuously* from some finite value to infinity and back again in a short regular period $\frac{\pi}{\omega}$. Thus R might be supposed to be $\rho \sec^2 \omega t$; and the equation would be

$$L\frac{di}{dt}+\rho i \sec^2 \omega t=E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

of which the solution is

$$i=Ce^{-\frac{\rho}{L}\tan\omega t}+\frac{E}{L}e^{+\frac{\rho}{L}\tan\omega t}\int e^{-\frac{\rho}{L}\tan\omega t}dt; \quad . \quad . \quad (10)$$

but the integral $\int e^{\tan x}dx$ does not appear to be evaluable except by an unmanageable lot of series.

Equation to an Alternating Current.

10. Instead of varying the resistance, we may produce an alternating current by a periodic electromotive force, as, for

instance, when the circuit contains an electromagnetic machine of constant resistance, such as a telephone itself. The electromotive force is then representable by a simple harmonic function or by a sum of a number of these. Take the simplest case,

$$\mathrm{L}\frac{di}{dt}+\mathrm{R}i=\mathrm{E}\sin 2\pi nt, \quad . \quad . \quad . \quad . \quad (11)$$

where the electromotive force oscillates from E to $-$E and back again n times a second. The solution of this is

$$i=\mathrm{C}e^{-\frac{\mathrm{R}}{\mathrm{L}}t}+\frac{\frac{\mathrm{R}}{\mathrm{L}}\sin 2\pi nt-2\pi n\cos 2\pi nt}{\frac{\mathrm{R}^2}{\mathrm{L}^2}+4\pi^2n^2}\cdot\frac{\mathrm{E}}{\mathrm{L}}. \quad . \quad (12)$$

The first term rapidly dies out; and so the permanent value is

$$i=\frac{\mathrm{E}}{\mathrm{R}}\cdot\frac{\sin\omega t-\lambda\omega\cos\omega t}{1+\lambda^2\omega^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

writing ω for $2\pi n$ and λ for the time-constant $\frac{\mathrm{L}}{\mathrm{R}}$.

On the Law of Variation of a Battery-current in a Polarizable Circuit.

11. In the previous sections we have considered the electromotive force E of the battery to be constant. Now this is never accurately true, as the electromotive force begins to fall off the instant the circuit is closed, and some discrepancies between theory and experiment may arise from this cause. Let us see what happens when the battery is not constant, or when a polarizable voltameter forms part of the circuit. According to Kohlrausch, the electromotive force of polarization is proportional to the amount of decomposition, and therefore a short time t after the current has been established it is

$$p\int_0^t idt,$$

where p is the electromotive force produced by the passage of a unit of electricity. The equation to the current is therefore

$$\mathrm{L}\frac{di}{dt}+\mathrm{R}i=\mathrm{E}-p\int_0^t idt,$$

or

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+pi=0; \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and the integral of this, remembering that $i=0$ when $t=0$, is

$$i=\frac{2E}{R}e^{-\alpha t}\sinh\beta t, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where

$$\alpha=\frac{R}{2L} \text{ and } \beta=\sqrt{\left(\frac{R^2}{4L^2}-\frac{p}{L}\right)}.$$

If $p=\frac{R^2}{4L}$, there could never be any current; and it is impossible for p to be greater than $\frac{R^2}{4L}$. Practically, however, p would always be very small compared with this quantity; and so we may write

$$\beta \doteq \frac{R}{2L}-\frac{p}{R}.$$

Hence in any ordinary polarizable circuit the strength of the current at any time after "make" is

$$i=\frac{E}{R}\left(e^{-\frac{p}{R}t}-e^{\left(\frac{p}{R}-\frac{R}{L}\right)t}\right). \quad . \quad . \quad . \quad . \quad (16)$$

On the Law of Variation of a Battery-current when a closed secondary circuit is stationary in its neighbourhood.

12. So far we have considered a primary circuit in space by itself; but now we will arrange near it a secondary coil with a resistance r and a coefficient of self-induction l, and we will suppose M to be the coefficient of mutual induction between the primary and secondary. The coils shall be fixed in position so that M is constant; then a current j will be developed in the secondary circuit whenever the primary changes from R to S; and this will react on the primary current i, the equations for determining both being

$$L\frac{di}{dt}+M\frac{dj}{dt}+Si=E \quad . \quad . \quad . \quad . \quad (17)$$

and

$$l\frac{dj}{dt}+M\frac{di}{dt}+rj=0, \quad . \quad . \quad . \quad . \quad (18)$$

with the conditions $i=\frac{E}{R}$ and $j=0$ when $t=0$.

Differentiating these (E being supposed constant), we easily separate the variables and obtain

$$(\mathrm{L}l-\mathrm{M}^2)\frac{d^2i}{dt^2}+(\mathrm{L}r+\mathrm{S}l)\frac{di}{dt}+\mathrm{S}ri=\mathrm{E}, \quad . \quad . \quad (19)$$

and

$$(\mathrm{L}l-\mathrm{M}^2)\frac{d^2j}{dt^2}+(\mathrm{L}r+\mathrm{S}l)\frac{dj}{dt}+\mathrm{S}rj=0. \quad . \quad . \quad (20)$$

Now the solution of (20), remembering the initial condition, is

$$j=\mathrm{J}e^{-\alpha t}\sinh\beta t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

or

$$=\tfrac{1}{2}\mathrm{J}\{e^{-(\alpha-\beta)t}-e^{-(\alpha+\beta)t}\},$$

where

$$\alpha=\frac{\mathrm{L}r+l\mathrm{S}}{2(\mathrm{L}l-\mathrm{M}^2)} \text{ and } \beta=\frac{\sqrt{\{(\mathrm{L}r-l\mathrm{S})^2+4r\mathrm{SM}^2\}}}{2(\mathrm{L}l-\mathrm{M}^2)} \quad . \quad (22)$$

$$\left[\text{i. e.} \quad \alpha^2-\beta^2=\frac{r\mathrm{S}}{\mathrm{L}l-\mathrm{M}^2}\right],$$

and where J is a constant which has yet to be determined. It is well to notice that neither α nor β can vanish or become imaginary; but they can become infinite simultaneously.

To obtain the value of the primary current i, we can either write down the solution of (19), viz.

$$i=\frac{\mathrm{E}}{\mathrm{S}}+e^{-\alpha t}(\mathrm{C}_1e^{\beta t}+\mathrm{C}_2e^{-\beta t}); \quad . \quad . \quad . \quad . \quad (23)$$

or, what is practically more convenient as determining all arbitrary constants directly, we can combine equations (17) and (18) so as to eliminate $\frac{di}{dt}$, and then put in the values of j and $\frac{dj}{dt}$ from (21). We thus get

$$\mathrm{M}(\mathrm{S}i-\mathrm{E})=\tfrac{1}{2}\mathrm{J}e^{-\alpha t}\{\sqrt{(\mathrm{L}r-\mathrm{S}l)^2+4r\mathrm{SM}^2}.\cosh\beta t$$
$$+(\mathrm{L}r-l\mathrm{S})\sinh\beta t\},$$

which will give us the value of J, since $i=\frac{\mathrm{E}}{\mathrm{R}}$ when $t=0$. We find then that

$$\mathrm{J}=\frac{2\mathrm{ME}(\mathrm{S}-\mathrm{R})}{\mathrm{R}\sqrt{\{(\mathrm{L}r-l\mathrm{S})^2+4r\mathrm{SM}^2\}}}; \quad . \quad . \quad . \quad . \quad (24)$$

and this is the value to be substituted in equation (21).

Hence, finally, we can write out explicitly the values of i and j, though, as the constants are very long, we will make an abbreviation by writing

$$\frac{4r\mathrm{SM}^2}{(\mathrm{L}r-l\mathrm{S})^2}=\lambda^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

then the strength of the battery-current at any time after the resistance of its circuit has suddenly jumped from R to S is

$$i=\frac{\mathrm{E}}{\mathrm{S}}+\frac{\mathrm{E(S-R)}}{\mathrm{SR}}e^{-\alpha t}\left\{\cosh\beta t+\frac{\sinh\beta t}{\sqrt{(1+\lambda^2)}}\right\}; \quad . \quad (26)$$

and the strength of the induced current at the same instant is

$$j=\frac{2\mathrm{ME(S-R)}}{\mathrm{R}(\mathrm{L}r-l\mathrm{S})\sqrt{(1+\lambda^2)}}e^{-\alpha t}\sinh\beta t. \quad . \quad . \quad . \quad (27)$$

We may notice that the expression for i contains only the *square* of M; that is, the square of M expresses the reaction of the secondary on the primary; hence when M is small this may be neglected.

Special Cases.

13. It will be interesting now to consider the special cases in which these equations (26) and (27) may be expected to assume simple forms.

Case 1. When the coils are so far apart that M^2 is negligible compared with $\mathrm{L}l$.

In this case $\lambda=0$, $\alpha+\beta=\frac{r}{l}$, $\alpha-\beta=\frac{\mathrm{S}}{\mathrm{L}}$; and it will be found that the expression for the primary current i as given in equation (26) reduces to the expression (4), which we found for the current in a solitary primary circuit, as it evidently ought to do; and there is nothing new to be said about it.

But the value of j, the current induced in the secondary circuit, becomes

$$j=\frac{\mathrm{ME(S-R)}}{\mathrm{R}(\mathrm{L}r-l\mathrm{S})}(e^{-\frac{\mathrm{S}}{\mathrm{L}}t}-e^{-\frac{r}{l}t}). \quad . \quad . \quad . \quad (28)$$

To find the induced current at instantaneous break in the special case now being considered, we must put $\mathrm{S}=\infty$, and we get

$$j=\frac{\mathrm{ME}}{l\mathrm{R}}e^{-\frac{r}{l}t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

which is the recognized value for it (see Chrystal, 'Encyc. Brit.' equation (41)). Its initial or maximum value is $\frac{\mathrm{M}}{l} \cdot \frac{\mathrm{E}}{\mathrm{R}}$; and it rapidly dies away.

To find the induced current at "make" we must put $\mathrm{R}=\infty$; and then (28) becomes

$$j=-\frac{\mathrm{ME}}{(\mathrm{L}r-l\mathrm{S})}\left(e^{-\frac{\mathrm{S}}{\mathrm{L}}t}-e^{-\frac{r}{l}t}\right). \quad . \quad . \quad . \quad (30)$$

This, therefore, begins at zero, rises to a maximum after the lapse of time

$$t=\frac{\mathrm{L}l}{\mathrm{L}r-l\mathrm{S}}\log\frac{\mathrm{L}r}{l\mathrm{S}},$$

and then dies away.

If the primary and secondary circuits are *similar*, so that

$$\frac{\mathrm{S}}{\mathrm{L}}=\frac{r}{l},$$

the expression for the induced current at make simplifies, becoming

$$j=-\frac{\mathrm{ME}}{\mathrm{L}l}te^{-\frac{r}{l}t}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

14. *Case* 2.—When the primary and secondary circuits are wound side by side, so that the coefficient of mutual induction nearly equals the coefficient of self-induction of either—in other words, so that $\mathrm{L}=l=\mathrm{M}$.

For this case (see equations 22 and 25),

$$\lambda=\frac{4r\mathrm{S}}{(r-\mathrm{S})^2}, \text{ and } \alpha=\beta=\infty;$$

but if we put $\mathrm{L}-\mathrm{M}=$ a small quantity z, say, and then proceed to the limit, we shall find a finite value for the difference of α and β, viz.

$$\alpha-\beta=\frac{r\mathrm{S}}{\mathrm{L}(r+\mathrm{S})},$$

while

$$\alpha+\beta=\frac{r+\mathrm{S}}{2z};$$

and accordingly we get as the value of the primary current (26),

$$i=\frac{\mathrm{E}}{\mathrm{S}}\left\{1+\frac{\mathrm{S}-\mathrm{R}}{\mathrm{R}(r+\mathrm{S})}\left\{re^{-\frac{r\mathrm{S}}{\mathrm{L}(r+\mathrm{S})}t}+\mathrm{S}e^{-\frac{r+\mathrm{S}}{2z}t}\right\}\right\}. \quad (32)$$

The last term in these brackets decreases at a nearly infinite rate; hence the primary current jumps almost suddenly from the value $\frac{E}{R}$ to the value

$$\frac{E}{R} \cdot \frac{r+R}{r+S}, \quad \ldots \ldots \quad (33)$$

and then increases (or decreases, as the case may be) at a more moderate pace, its subsequent values being given by the equation

$$i = \frac{E}{S}\left(1 + \frac{r(S-R)}{R(r+S)} e^{-\frac{rS}{L(r+S)}t}\right). \quad \ldots \quad (34)$$

The form of expression (32) when S is made very great is noticeable, as it shows that the primary current in a wire coiled up with a closed secondary is able to stop nearly as dead as if the primary were doubled upon itself (6). It is

$$i = \frac{E}{R} e^{-\frac{S}{2z}t}. \quad \ldots \ldots \quad (35)$$

The value of the induced current under the same circumstances (viz. when the two circuits are coiled close together) is easily obtained from (27), and is

$$j = \frac{E(S-R)}{R(r+S)} e^{-\frac{rS}{l(r+S)}t}. \quad \ldots \ldots \quad (36)$$

If $S = \infty$, this is the current at instantaneous break, viz.

$$j = \frac{E}{R} e^{-\frac{r}{l}t}, \quad \ldots \ldots \quad (37)$$

the same as (29) would have given; but if $R = \infty$, it is the current at make, viz.

$$j = -\frac{E}{r+S} e^{-\frac{S}{r+S} \cdot \frac{r}{l}t}. \quad \ldots \ldots \quad (38)$$

The induced current at *make* in this case therefore has an instantaneous maximum value $\frac{E}{r+S}$ and it then dies away at about half the rate of the current at break.

All this is very instructive. It must be the state of things approximated to in many arrangements.

15. We can now see what happens in Mr. Grant's case of a primary wound alongside a secondary, whose circuit can be

closed or unclosed; though the state of things considered in the last section is only roughly attained, because the telephone-coil forms part of the battery-circuit, and adds materially to the self-induction of that circuit without contributing to the mutual induction. However, neglecting this and taking the circuits similar, we find the primary current at "make" ($R=\infty$, $S=r$), by equation (32), to be

$$\frac{E}{S}\left(1-\tfrac{1}{2}e^{-\frac{S}{2L}t}-\tfrac{1}{2}e^{-\frac{S}{\epsilon}t}\right),$$

whereas if the secondary circuit had been unclosed it would have been (5)

$$\frac{E}{S}\left(1-e^{-\frac{S}{L}t}\right).$$

Similarly, at the break ($S=$ large, $R=r$), the primary current, when the secondary is closed, is

$$\frac{E}{R}e^{-\frac{S}{2\epsilon}t}$$

instead of

$$\frac{E}{R}e^{-\frac{S}{L}t}$$

when it was open.

Hence the initial or maximum rate of variation of the battery-current either at make or break when the secondary current is closed, is to its initial rate of variation when the secondary is unclosed as L is to $2(L-M)$,—a ratio which, in the case supposed, is nearly infinite.

The Theory of Tertiary Currents as produced in an induction-balance—that is, in a coil which is arranged so as to be conjugate to the primary coil, but which is in the neighbourhood of a third coil or conductor affected inductively by the primary current.

16. This is the case of the induction-balance disturbed by a coin. The primary and secondary circuits are arranged in some conjugate position so that they have no effect on each other—this being done practically by splitting each into two equal parts, and arranging the two parts so as to oppose each other; and the third circuit is represented by the piece of metal put near or inside the primary and secondary coils.

The primary circuit contains as part of itself a battery and an intermitter; the secondary contains a telephone.

Neglecting electrostatic capacity and leakage (which latter, by the way, must be carefully avoided in a good balance), the current in either primary is the same, and the current in each secondary is the same, not only on the whole, but at every instant of time. In fact, the separation of the two halves of the primary and secondary is immaterial to the theory. The current which is induced in the secondary circuit, and which is heard in the telephone, is a tertiary current produced by induction from the third coil or piece of conducting matter.

The currents in the battery-circuit at any instant being called i, that induced in the metal being k, and that induced by this in the secondary circuit being called j, and being the value appreciated by the telephone, the equations for determining these three quantities are given in § 5, on the supposition that the coefficients of mutual induction between the metal and the two coils respectively (m and μ) are small numbers, which is always true in practice.

The battery-circuit is then practically undisturbed by the presence of the other parts of the instrument, and may be regarded as in space by itself (see end of §§ 12 and 13); so that all we have said about the intermittence of currents in such a circuit in sects. 6–11 applies to the primary of an induction-balance: we will, however, only take the case when the resistance jumps suddenly from any finite value R to another finite value S. The value of i is then given by equation (4); and its rate of variation is

$$\frac{di}{dt} = -\frac{\mathrm{E(S-R)}}{\mathrm{RL}} e^{-\frac{\mathrm{S}}{\mathrm{L}}t};$$

but as (equation 2)

$$\lambda \frac{dk}{dt} + \rho k + \mu \frac{di}{dt} = 0,$$

it is easy to determine k; viz.

$$k = \frac{\mu \mathrm{E(S-R)}}{\mathrm{R(L}\rho - \mathrm{S}\lambda)} \left\{ e^{-\frac{\mathrm{S}}{\mathrm{L}}t} - e^{-\frac{\rho}{\lambda}t} \right\}. \quad . \quad . \quad (39)$$

This value agrees exactly with that obtained in the general case (§ 12) when the condition that M^2 might be neglected was put in (see eq. 28), and therefore confirms the correctness

of our working. All that was said about the special cases of j in § 13 is therefore now true of k. What we want to find, however, is not k but j, the current induced in the telephonic circuit; hence, differentiating the above expression for k and substituting it in

$$l\frac{dj}{dt}+rj+m\frac{dk}{dt}=0,$$

we get, remembering the initial condition that $j=0$ when $t=0$,

$$j=-\frac{m\mu\mathrm{E}(\mathrm{S}-\mathrm{R})}{\mathrm{R}}\left\{\frac{r}{(\mathrm{L}r-l\mathrm{S})(l\rho-\lambda r)}e^{-\frac{r}{l}t}\right.$$
$$\left.+\frac{\rho}{(l\rho-\lambda r)(\lambda\mathrm{S}-\mathrm{L}\rho)}e^{-\frac{\rho}{\lambda}t}+\frac{\mathrm{S}}{(\lambda\mathrm{S}-\mathrm{L}\rho)(\mathrm{L}r-l\mathrm{S})}e^{-\frac{\mathrm{S}}{\mathrm{L}}t}\right\}. \quad (40)$$

And this is the current heard in the telephone.

The expression $\int_0^\infty jdt$ identically vanishes; hence this current cannot affect a galvanometer (see § 1).

At "make" $\mathrm{R}=\infty$, and the three terms in the brackets are all present, the factor outside reducing to $+m\mu\mathrm{E}$. But at "break" $\mathrm{S}=\infty$, and the third term disappears, leaving

$$j=\frac{m\mu\mathrm{E}}{\mathrm{R}(l\rho-\lambda r)}\left\{\frac{r}{l}e^{-\frac{r}{l}t}-\frac{\rho}{\lambda}e^{-\frac{\rho}{\lambda}t}\right\}; \quad . \quad . \quad (41)$$

which is the tertiary current at break, and has some of the characteristics of the secondary current at make, see equation (28).

17. If the battery and telephone circuits are *similar*, so that $\frac{\mathrm{S}}{\mathrm{L}}=\frac{r}{l}$, the value of j at the make simplifies considerably, becoming

$$\frac{m\mu\mathrm{ES}}{\mathrm{L}l(\lambda\mathrm{S}-\mathrm{L}\rho)}\cdot te^{-\frac{r}{l}t}\Big\}, \quad . \quad . \quad . \quad . \quad (42)$$

which in this case therefore bears a constant ratio to the secondary current at make under the same circumstances (see equation 31).

Theory of the Induction-balance continued.

18. Having now integrated equations (2) and obtained an expression (40) for the current passing through the telephone at any instant, several things may be noticed about it. First,

the current produced by the piece of metal is in general different in character to that produced by a slight shift of the secondary coil (that is, by M being made not quite zero); and consequently it is impossible in general to balance the effect of metal completely by moving the other pair of coils a little further apart, though it may be done partially. Both effects would then be superposed, and the two sounds would be separately heard and might differ in pitch, because there would be more oscillations in the tertiary current than in the secondary. It is quite possible that a musical ear would perceive that the quasi-note produced by the insertion of a piece of metal contains a trace of a tone an octave above that produced by a shift of the coils. Equation (42), however, appears to show that when the battery- and telephone-circuits are *similar* the secondary and tertiary currents are expressible by the same function of the time, and that, therefore, in this case the effect of introducing metal *may* be counteracted by a shifting of the coils.

Next, we see that the effect is independent of M (that is, of the mutual induction between the primary and secondary coils); and therefore it may occasionally be well to place them in some nearly conjugate position, so as more easily to obtain a balance. In fact the four coils are unnecessary; two coils placed so as not to act on each other will do; and a piece of metal placed near them will produce disturbance; only it will not be so easy to bring the metal close or parallel to both coils.

Next the effect is proportional to the produced $m\mu$; of which μ is the potential of the primary, and m that of the secondary, on the coin. Hence, if the two coils are equal in size, every thing is symmetrical with respect to them, so that it does not matter whether the coin is nearest the primary or the secondary; moreover the middle point halfway between the two coils must be either a maximum or a minimum position of the coin.

Prof. Hughes long ago stated, from experimental observation, that the position of the coin which gave the loudest sound was half way between primary and secondary, the two coils being pretty close together and being equal in all respects. Before investigation I had felt inclined to doubt whether this were more than approximately true (as it was by no means

evident *à priori* that every thing was symmetrical with respect to both coils, and the middle of the primary coil seemed a not improbable place for the maximum). Accordingly I arranged two pairs of coils as an induction-balance, the components of each pair being 4 or 5 inches apart, but with their planes parallel as usual; and I then moved a coin about along the common axis of one pair to see what happened. The maximum was *not* in the middle; but there were two maxima, one on each side the middle, and every thing appeared symmetrical with respect to both coils. The preceding investigation throws light upon this, and reconciles the two observations. It turns out (as might have been anticipated) that Professor Hughes's statement is quite correct when the coils are near together; but if they are separated by a distance greater than the diameter of either (the coin being supposed small), the middle point becomes a *minimum* with a maximum on each side of it (see next section).

On the Law according to which the disturbance produced by a small coin on the common axis of a pair of coils in an induction-balance depends on its position.

19. The general expression for the induction-coefficient between two circular coils in any position whatever is given in 'Maxwell,' art. 696. We can specialize this to the case required, viz. a coil of n turns and a "coin" or coil of one turn, both on the same axis. Let the mean radius of the "coil" be a, and let the distance of its circumference from some point O on the axis be c. Also let the mean radius of the "coin" be b, and let d be the distance of its centre from the point O. Then the mutual induction-coefficient between the coil and the coin is (art. 699)

$$2n\pi^2\frac{a^2b^2}{c^3}\left\{1+3\frac{\sqrt{(c^2-a^2)}}{c^2}d+6\frac{c^2-\frac{5}{4}a^2}{c^4}(d^2-\tfrac{1}{4}b^2)+\&c.\right\}.$$

Now d we can make zero at once by taking the point O at the middle of the coin, so that c is the distance from the mean circumference of the coil to the middle of the coin; and there remain inside the brackets of the above expression 1 + terms involving the square and higher powers of $b:c$, which is a small quantity. Thus a very good approximation to the in-

duction-coefficient between the primary and a small coin is

$$\mu = \frac{2\pi^2 a^2 b^2 n}{c^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

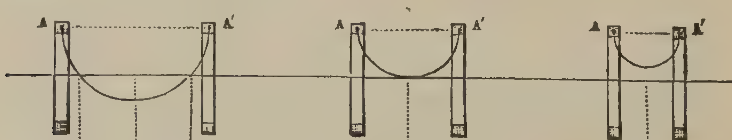
Similarly the induction-coefficient between the secondary and the coin is the same expression with dashed letters ; hence the product

$$m\mu = \frac{nn'}{(cc')^3}(2\pi^2 b^2 aa')^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

And this is a maximum when cc' is a minimum, a thing which is very easily represented geometrically.

For let A and A′ be points on the mean circumferences of the coils from which c and c' are measured; draw the system of lemniscates cc' = various constants, and then draw through the system any straight line at a distance a from A and a' from A′. This line will represent the common axis of the two coils, and points on it represent possible positions of the coin. Now the line cuts every lemniscate in four points: hence there are four points on the axis at which the coin produces the same disturbance. These can run together two and two at points where the line is touched by the lemniscate; and these are the maximum or minimum points. They are maxima if the line is touched on the side of A A′, but minima if the line is touched on the other side.

If the coils are the same size (*i. e.* if $a = a'$), and if they are at a proper distance apart (viz. $= 2a$), all four points can run together in the middle, which is therefore then a stationary point, so that moving the coin a little either way will produce very little diference. But if the coils are closer together than this, two of the points of section become imaginary, and hence there are now only two positions of the coin which give the same noise; and these two can run together in the middle, which is now a maximum.



Analytically the problem might be stated thus:—Find the max. and min. values of $\sin\theta \sin\theta'$, given the condition

$a \cot \theta + a' \cot \theta' = \text{const.}$, θ and θ' being the angles subtended by mean radii of the primary and secondary coils at the coin. If the coils are equal ($a = a'$), the solutions are $\theta = \theta'$ and $\theta + \theta' = \frac{1}{2}\pi$; hence another way of stating the result is the following:—Draw a circle with A A′ (the mean points of the equal coils) as diameter: then if this circle cuts the common axis of the coils, the points of section are maxima, and the middle point is a minimum; but if the circle does not cut the axis, the middle point is a maximum, and the only one (see figure).

The law of decrease with distance is interesting: equation (44) shows us that when the two coils are close together and the coin is moved along the axis away from them, the effect which it is able to produce in the telephone varies pretty nearly as the inverse *sixth* power of the distance as soon as it has got a little way—a tremendously rapid rate of decrease.

It also shows that the effect varies directly with the fourth power of the diameter of the coin.

The Connexion between Induction-balance Effect and Conductivity.

20. The way in which the telephone-current depends on the conductivity $\frac{1}{\rho}$ of the coin is apparent in equation (40): it is evidently not simply proportional to it in general. But the self-induction λ of a circuit like that in a solid disk of metal must always be a very small quantity; and, except perhaps for highly conducting metals, it must probably be almost negligible compared with ρ. Assuming, then, that λ is infinitesimal, the expression simplifies, and ρ becomes a factor of one part of it. We get, in fact,

$$j = \frac{m\mu \mathrm{E}(\mathrm{S}-\mathrm{R})}{\rho \mathrm{R}(\mathrm{L}r - l\mathrm{S})} \left\{ \frac{\mathrm{S}}{\mathrm{L}} e^{-\frac{\mathrm{S}}{\mathrm{L}}t} - \frac{r}{l} e^{-\frac{r}{l}t} \right\} + \frac{m\mu \mathrm{E}(\mathrm{S}-\mathrm{R})}{\mathrm{R}(\mathrm{L}-l)} e^{-\frac{\rho}{\lambda}t} \quad . \quad . \quad . \quad . \quad (45)$$

The term written by itself does not contain ρ except as an exponential; but it has a very great rate of variation depending directly on ρ, and it soon ceases to exist. The rest of the expression for the current is simply proportional to the con-

ductivity, both as regards its own value and its rate of variation. What the precise meaning of all this is depends on what the loudness of the telephone-indication definitely depends on (§ 1); so for the present we will leave it in this state.

On Measuring with the Induction-balance.

21. Two methods have been employed both by Professor Hughes and by Mr. Roberts, with the view of obtaining quantitative readings from the balance. The first consisted in estimating the loudness of the sound produced by the insertion of a coin, and then imitating it by an arrangement of primary and secondary coils called the "sonometer"—a key being used to transfer the connexions of the telephone quickly from one instrument to the other. Mr. Poynting has adapted a formula from Maxwell which suffices to graduate the sonometer when the secondary coil is some distance from either primary (Phil. Mag. January), though it is rather an unwieldy one.

But there must always be some objection to readings taken in this way, because of the difficulty of estimating precisely when two sounds have the same strength—especially if, as appears probable from § 18, they differ at all in quality, the sound produced by the coin in the balance having a tone mixed up with it which is shriller than any produced by the sonometer. (I do not know whether this has ever been noticed experimentally. One often notices a change of pitch in the telephone-rustle; but it appears to depend in some cases on the direction of the current, *i. e.* on whether it strengthens or weakens the magnet.)

If the sonometer is to be used, I would suggest that the second primary coil be done away with, and that the secondary coil be of small diameter. It may have as much wire on it as is wanted for long range; but the further it is away from the primary, compared with its own diameter, the better. The diameter of the primary does not much matter. The induction would now simply vary as the inverse cube of the distance from the mean circumference of the primary; hence, if the sonometer were graduated so as to give equal distances from this line, its readings would only have to be turned upside down and cubed to give comparative results. If both

primary and secondary coils are small, the sonometer-arm may be graduated like an ordinary millimetre-scale.

If an absolute zero were wanted at any time, it might be obtained by an arrangement for rotating one of the coils till it was at right angles to the other.

22. Prof. Hughes's second method of measuring, that of the graduated zinc wedge, is a much better one, because it is a null method and gives true readings, though they are not easily interpretable.

A third method has, I believe, been tried, viz. a copper damper rotating above one pair of coils; and this seems also pretty good; but I think it might possibly be better to modify it by having a uniform disk (or ring) of high-conductivity copper capable of being moved along the axis of one pair of coils with its plane always parallel to theirs. The distance of the disk from the middle point of the coils should be read, and the distance apart of the coils should be constant. Given all particulars, I have no doubt that a mathematician could readily interpret results so obtained.

Another modification would be to use a coil of wire forming a circuit closed within itself instead of the copper disk. It would have just about the same effect as a solid disk of the same size; but the data connected with it might be more definite.

The Effect of Magnetic Bodies.

23. If a thin piece of iron wire be held in the balance, it powerfully disturbs it by a concentration of the lines of force in the iron, so that M (the mutual induction between the battery- and telephone-circuits) is no longer zero. If the iron were instantaneously magnetizable, one ought to be able to balance its effect by moving the other pair of coils a little nearer together, so as to make M zero again. But as the magnetization of iron always takes time, this is not completely possible, because, though M may be made zero at any one instant, it will not remain so during the period of variation of the current. As regards pitch, the rustle produced by iron wire should agree with that produced by a shift of the coils, both being duller than that produced by a piece of copper. Of course if a *mass* of iron be inserted, the effect will be compo-

site, secondary currents being induced in it which give rise to tertiary currents in the telephone-circuit. But this effect is feeble as compared with the magnetic effect.

In measuring the conductivity of a metal or of an electrolytic solution by means of the induction-balance, care must be taken that it is not magnetic; or erroneous results will be obtained. The conductivity effect and the magnetic effect will tend partially to destroy each other; for though the effect of a piece of copper cannot be completely balanced by introducing a scrap of iron into the same pair of coils, it can partially, just as it might partially by shifting the coils (§ 18).

It would be very interesting to observe an opposite effect with a diamagnetic body—say a large bundle of thin bismuth wires insulated from each other; but it would not be easy to make sure of the correctness of the observation.